An Alternative Method for Computing General Equilibrium Models

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Summary

- We ask "what-if" questions in GE trade literature.
- There are many similar tools to do this, e.g., Markusen (1984), Rutherford (1995), Balistreri and Hillberry (2007), Deckle, Eaton and Kortum (2008), Arkolakis, Costinot and Rodríguez-Clare (2012), and the GTAP/other CGE models largely expanded since 1990s'.
- If the model is getting larger and more complex, e.g., implicit models, then some of the tools may be hard to solve.
- We review and compare different approaches, e.g., exact hat algebra and MCP methods of counterfactual computation.
- We discuss another computation method in Yang (2020) for answering "what-if" questions.
- We bridge exact hat algebra with MCP approach, then with the new method.

Exact Hat Algebra

- We choose the CES utility function:

$$U_j = \left[\sum_{i}^{n} \beta_i^{\frac{1}{\sigma}} T_{ij}^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \tag{1}$$

- Trade share of i's goods in j is given by:

$$\pi_{ij} \equiv \frac{X_{ij}}{Y_j} = \beta_i \left(\frac{p_{ij}}{P_j} \right)^{1-\sigma} \tag{2}$$

- CES price index (unit expenditure function):

$$P_j = \left[\sum_{i}^{n} \beta_i p_{ij}^{1-\sigma} \right]^{\frac{1}{1-\sigma}} \tag{3}$$

Exact Hat Algebra

- We consider a counterfactual "what-if" scenario: policy instrument affecting trade costs: $\boldsymbol{\tau} \equiv [\tau_{ij}] \longrightarrow \boldsymbol{\tau'} \equiv [\tau'_{ij}]$
- For any generic variable V in the baseline equilibrium, V' is the value in the new equilibrium, and $\widehat{V} = V'/V$.
- The trade share is simplified as:

$$\pi_{ij} = \frac{\beta_i \tau_{ij}^{1-\sigma} FOB_i^{1-\sigma}}{\sum_k^n \beta_k \tau_{kj}^{1-\sigma} FOB_k^{1-\sigma}} \tag{4}$$

- Given the income definition:

$$Y_i = FOB_i e_i^0 \tag{5}$$

- We define $\delta_i \equiv \beta_i (e_i^0)^{\sigma-1}$, with e_i^0 which can be understood as i's endowment quantity.

Exact Hat Algebra

- The initial equilibrium is then given by:

$$\pi_{ij} = \frac{\delta_i(\tau_{ij} Y_i)^{1-\sigma}}{\sum_k^n \delta_k(\tau_{kj} Y_k)^{1-\sigma}} \tag{6}$$

- Note that Eq. (6) essentially gets rid of local price of endowment units which we do not have good price data for. δ_i is later also eliminated, so we do not need to estimate β_i.
- In counterfactual equilibrium:

$$\pi'_{ij} = \frac{\delta_i(\tau'_{ij} Y'_i)^{1-\sigma}}{\sum_k^n \delta_k(\tau_{kj} Y'_k)^{1-\sigma}} \tag{7}$$

- Eqs.(6) and (7) deliver

$$\hat{\pi}_{ij} = \frac{(\hat{\tau}_{ij} \hat{Y}_i)^{1-\sigma}}{\sum_k^n \pi_{kj}(\hat{\tau}_{kj} \hat{Y}_k)^{1-\sigma}} \tag{8}$$

Exact Hat Algebra

- The goods-market clearing conditions are

$$Y_i = \sum_j^n \pi_{ij} Y_j \tag{9}$$

$$Y_i' = \sum_j^n \pi_{ij}' Y_j' \tag{10}$$

- This gives:

$$\hat{Y}_i Y_i = \sum_i^n \hat{\pi}_{ij} \pi_{ij} \hat{Y}_j Y_j \tag{11}$$

Exact Hat Algebra

- Substituting Eq.(8) in (11) gives us

$$\hat{Y}_i Y_i = \sum_{i}^{n} \frac{(\hat{\tau}_{ij}\hat{Y}_i)^{1-\sigma}}{\sum_k^n \pi_{kj}(\hat{\tau}_{kj}\hat{Y}_k)^{1-\sigma}} \pi_{ij}\hat{Y}_j Y_j \tag{12}$$

- Y_j and π_{ij} are data; $\hat{\tau}_{ij}$ are "what-if" known by us.
- Since there are N equations and N unknowns, we can solve the system above, finding $\hat{Y}_i$.
- Taking $\hat{Y}_i$ to Eq.(8), we can solve for $\hat{\pi}_{ij}$.
- Estimating σ without over-restricted normalization is difficult.
- From the econometric perspective, it is challenging to jointly identify U at initial equilibrium and σ.
- A common practice is to choose σ.

Exact Hat Algebra

- Moving to solve for welfare change, we get rid of the local price term and use what we have from the data:

$$P_j^{1-\sigma} = \sum_i^n \beta_i p_{ij}^{1-\sigma} = \sum_i^n \delta_i (\tau_{ij} Y_i)^{1-\sigma} \tag{13}$$

- The welfare definition or income balance condition:

$$Y_j = U_j P_j \tag{14}$$

- We have country j's share of own consumption:

$$\pi_{jj} = \frac{\delta_j \tau_{jj}^{1-\sigma} Y_j^{1-\sigma}}{P_j^{1-\sigma}} = \delta_j \tau_{jj}^{1-\sigma} U_j^{1-\sigma} \tag{15}$$

Exact Hat Algebra

- Therefore,

$$U_j = \left(\frac{\pi_{jj}}{\beta_j \tau_{jj}^{1-\sigma}} \right)^{\frac{1}{1-\sigma}} \tag{16}$$

- β_j is exogenous, $\tau_{jj} = 1$.
- Hence,

$$\hat{U}_j = (\hat{\pi}_{jj})^{\frac{1}{1-\sigma}} \tag{17}$$

- The welfare prediction depends on **only two sufficient statistics:** (1) share of expenditure on domestic goods; (2) elasticity of substitution (and can be generalize to trade elasticity under other trade-cost specification).
- This is, in fact, the ACR result (Arkolakis, Costinot and Rodríguez-Clare, 2012).

Mixed Complementarity Problems

- The GE conditions can be formulated as mixed complementarity problems (MCP).
- This approach requires us to first "know" β's (but not really).
- We also need to choose σ in order to estimate β's.
- The procedure identifies parameters at initial equilibrium by making the benchmark price normalization explicit:

$$FOB_i = 1 \quad \forall i \tag{18}$$

- This essentially chooses the units of local endowment e_i^0.
- From the exact hat algebra, we know that the price of endowment unit and quantity are irrelevant in determining counterfactual equilibrium.

Mixed Complementarity Problems

- This implies that the outcome of β's identified by any arbitrary choice of local price and endowment units does not affect counterfactual equilibrium (under the same GE specification).
- This can be verified computationally by shifting the FOB.
- Note that the numeraire price needs to be set consistently.
- Choosing the estimator, e.g., PPML, Least-Squares, subject to the same general equilibrium conditions.
- CES price index (unit expenditure function):

$$\Big[\sum_{i}^{n} \beta_i (FOB_i \tau_{ij})^{1-\sigma}\Big]^{\frac{1}{1-\sigma}} \geq P_j \quad \perp \quad U_j \geq 0$$

Mixed Complementarity Problems

- The explicit market-clearing condition can be solved by combining Eq. (2) and (9):

$$e_i^0 \geq \sum_j^n \left[\beta_i \frac{Y_j}{FOB_i} \left(\frac{FOB_i \tau_{ij}}{P_j} \right)^{1-\sigma} \right] \quad \perp \quad FOB_i \geq 0 \tag{19}$$

- The income balance condition:

$$U_j P_j \geq Y_j \quad \perp \quad P_j \geq 0 \tag{20}$$

- The income definition:

$$Y_i = FOB_i e_i^0 \tag{21}$$

Mixed Complementarity Problems

- The system formulated above can be solved in a non-linear program, such as NLP or mathematical programming with equilibrium constraints (MPEC).
- Fix the solved parameters, i.e., $\beta_i \longrightarrow$ data.
- Free income and price: data $\longrightarrow$ variables.
- This allows endogenous mechanism to determine counterfactual equilibrium.
- Implement policy instrument, i.e., $\tau_{ij} = 1 \longrightarrow \tau_{ij} = 1.25$.
- Identify complementarity variables.
- Solve for counterfactual equilibrium.

EMCP versus Exact Hat

- We may denote the full computation procedure as an EMCP approach (Estimation and MCP).
- Using Anderson and van Wincoop (2003) data as an exercise.
- It can be verified that the results solved using the EMCP are equivalent to the ACR and exact hat results.
- Both methods “ignore” β’s, benchmark prices and endowment quantities in some way.
- Exact hat elegantly eliminates β’s and prices directly, but one has to derive the counterfactual formulae.
- This can be quite tedious, and sometimes challenging if choosing more flexible models.
- EMCP approach does “estimate” β’s, but does so by choosing the units and prices.
- The choice of prices, e.g., setting to unity, does impact the outcome of $\beta's$, but not the counterfactual equilibrium.

EMCP versus Exact Hat

- This has some sense of "estibration" (Balistreri and Hillberry, 2005), while not fully estibrate, but does "calibrate" model parameters to one benchmark point, by estimating them using benchmark data and the choice of other exogenous information, that is not readily accessible, such that the general equilibrium system is fully operationalized.
- Because the estimation does all the hard work, we do not have to algebraically invert the model to calibrate the function coefficients as some work done in the CGE literature, e.g., "calibrated share form", Rutherford (1995).
- Both EMCP has some challenges, when choosing more flexible models. Implicit models are some good examples.

An Alternative Method

- Derivation of implicit models using exact hat can be difficult:

$$G = \sum_i \beta_i U_j^{e_i(1-\alpha_i)} (\frac{L_j FOB_i}{Y_j})^{1-\alpha_i} \equiv 1. \tag{22}$$

- In this case, Hanoch (1975), utility cannot be isolated and there is larger parameter space.
- Furthermore, it is difficult to identify complementarity variables because, for example, one single derived functional form represents both income balance condition and unit expenditure function:

$$P_j = \frac{\left[\sum_i \beta_i U_j^{e_i - e_i\alpha_i - 1} (1-\alpha_i) FOB_i^{1-\alpha_i} (Y_j/L_j)^{\alpha_i - 1} e_i\right]}{\left[\sum_k \beta_k U_j^{e_k(1-\alpha_k)} (1-\alpha_k) FOB_k^{1-\alpha_k} (Y_j/L_j)^{\alpha_k - 2}\right]} \tag{23}$$

An Alternative Method

- If we parameterize the system to a CES function, then Eq. (23) will collapse to the income balance condition, but not the unit expenditure function.
- This also makes it puzzling to solve using the conventional MCP approach.
- Alternatively, Yang (2020) uses an estimation approach based on the MPEC algorithm, and computes for both model parameters and counterfactuals as in the EMCP.
- It takes two solves, one to “estibrate” model parameters that are identified by the price normalization scheme and the data, to the benchmark equilibrium, while other parameters identified invariant to normalizing constants.
- The second solve takes care of the counterfactual calculation.

An Alternative Method

- The MPEC embeds an MCP, so it is suitable to estimate the general equilibrium relationship.
- It solves the problem by treating a parametric derived from the NLP problem fixed, while setting initial benchmark by constraining the likelihood of the objective function, such that $\overline{\mathscr{J}}(\overline{\mathscr{F}}, \overline{\boldsymbol{\alpha}}, \overline{\boldsymbol{e}}) \equiv \mathscr{J}^0(\overline{\mathscr{F}}, \overline{\boldsymbol{\alpha}}, \overline{\boldsymbol{e}})$ with the choice of numeraire
- $\overline{\mathscr{J}}$ is the estimated value of the objective function.
- While the complementarity theory is a discipline of mathematical optimization, the MCP does not carry a visible objective function.
- Nonetheless, shifting independent variables before and after the counterfactual equilibrium in fitting to the same data on dependent variable must have different likelihood values.

An Alternative Method

- The procedure essentially calibrates to the benchmark model by calibrating to the likelihood at initial equilibrium.
- In the second solve, release its restriction to the likelihood, while freeing model variables, compute the counterfactual equilibrium directly after the exogenous shocks.
- $\mathscr{J}^0(\overline{\mathscr{F}}, \overline{\boldsymbol{\alpha}}, \overline{\boldsymbol{e}}) \longrightarrow \mathscr{J}^1(\mathscr{F}, \overline{\boldsymbol{\alpha}}, \overline{\boldsymbol{e}})$
- data $\longrightarrow$ variables
- $\boldsymbol{\tau} \equiv [\tau_{ij}] \longrightarrow \boldsymbol{\tau'} \equiv [\tau'_{ij}]$
- It can be shown that results are equivalent to MCP and DEK.

An Alternative Method

- Why do we do want to do this?
- Economic models are becoming more and more complex.
- Computational expense, however, is becoming lower.
- Not every model is (easily) solvable by hand.
- Not every model has a clear intuition on complementarities.
- Not everyone needs to "fit the data to (large) models".
- It should really be the other way around.
- In this approach, one only needs to know the primal functional form of GE conditions.
- Other approaches rely largely on computations anyway.
- If we can show that the results with relatively less efforts can produce the same results, and are consistent with the theory, we should considering using it.