Build and Solve a Simple General Equilibrium (GE) Model with Hicksian LES Demand using a Mixed Complementarity Problem (MCP)

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Lecture developed for graduate international trade curriculum
Link to download code (with iteration)

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Exercises

- 2 representative firms, 2 representative consumers, 1 factor.
- Firms' technologies are linear with constant returns to scale.
- Consumers are modeled with linear expenditure system (LES);
- It can be easily parameterized to more general systems.
- See, e.g., Universal CES Demands.

Task

Build a GE model and formulate the economy as a mathematical programming problem using an MCP, and solve it.

Model Setup

- Consider an economy with two firms and two consumers.
- Firm 1 is entirely owned by consumer 1. It produces guns (g) from oil (x) via the production function:

$$g = 2x. \tag{1}$$

- Firm 2 is entirely owned by consumer 2; it produces butter (b) from oil via the production function:

$$b = 3x. \tag{2}$$

Model Setup

- Each consumer owns 10 units of oil.
- Consumer 1's utility function is

$$U(g, b) = g^{0.4} b^{0.6}. \tag{3}$$

- Consumer 2's utility function is

$$U(g, b) = 10 + 0.5 \ln g + 0.5 \ln b \tag{4}$$

Mechanical Solution

- Solving a GE first by hand can be handy for **several reasons**.
- First, ask **three questions**: (1) GE prices; (2) GE quantities of final goods; (3) GE input demand.
- *What are the market clearing prices for guns butter and oil*?
- Firms maximize their profits, firm 1

$$\begin{aligned} \underset{g,x\geq 0}{\text{maximizes}} \quad & \pi_1 = p_g g - p_x x \\ \text{s.t.:} \quad & g = 2x; \end{aligned} \tag{5}$$

- Isomorphically, firm 2

$$\begin{aligned} \underset{g,x\geq 0}{\text{maximizes}} \quad & \pi_2 = p_b b - p_x x \\ \text{s.t.:} \quad & b = 3x \end{aligned} \tag{6}$$

Mechanical Solution

- Both firms have production technologies that are linear with constant returns to scale (CRS), so are subject to zero profit in equilibrium.
- F.O.C. solutions with choice of numeraire: $p_x = 1$.
- $p_g = 1/2$ and $p_b = 1/3$.

Mechanical Solution

- *How many guns and how much butter does each consumer consume*?
- Both utility functions are Cobb-Douglas with convex optimization (concave objectives with linear constraints).
- So the optimal choices are share-weighted incomes divided by market-clearing prices derived earlier.
- $Q_{g,1} = 0.4 * 10 * 2 = 8,\ Q_{b,2} = 0.6 * 10 * 3 = 18.$
- $Q_{g,2} = 0.5 * 10 * 2 = 10,\ Q_{b,2} = 0.5 * 10 * 3 = 15.$

Mechanical Solution

- *How much oil does each firm use*?
- In market equilibrium, total output for guns and butter equals total consumptions that are produced by firm 1 and firm 2.
- That is, 18 units of gun, and 33 units of butter.
- Plugging them back into each firm's production function, we get the input requirements.
- $Q_{x,1} = 9$ and $Q_{x,2} = 11$.

Mechanical Solution

Consider a closed-economy GE framework with one intersectorally mobile factor L and two output (G and S).

The Stone-Geary form (LES): $e(\vec{p}, u) = p_G \bar{G} + U p_G^{\alpha} p_S^{1-\alpha}$,

where $\bar{G}$ in this representation is a "subsistence" or minimum level of goods consumption.

- One may use Shephard's Lemma to derive the Hicksian demand functions for G and S.
- **The Hicksian demand:** $h(\vec{p}, u) = \nabla_p e(\vec{p}, u)$.
- $X_G(\vec{p}, w) \equiv h_G(\vec{p}, u) = \bar{G} + \alpha U p_G^{\alpha-1} p_S^{1-\alpha}$.
- $X_S(\vec{p}, w) \equiv h_S(\vec{p}, u) = (1-\alpha) U p_G^{\alpha} p_S^{-\alpha}$

Dual Relationship and Roy's Identity

If consumers were given an additional dollar of income, how would they allocate it across the two goods?

- By the duality theory and money metric utility identity:

$$U \equiv V(\vec{p}, e(\vec{p}, U)) = V(\vec{p}, w) = \frac{w - p_G \bar{G}}{p_G^{\alpha} p_S^{1-\alpha}}. \tag{7}$$

- Using Roy's Identity:

$$\begin{aligned} x_G &= -\frac{\partial V / \partial p_G}{\partial V / \partial w} = (1-\alpha)\bar{G} + \alpha \frac{w}{p_G} \\ x_S &= -\frac{\partial V / \partial p_S}{\partial V / \partial w} = (\alpha - 1)\frac{p_G}{p_S}\bar{G} + (1-\alpha)\frac{w}{p_S}. \end{aligned} \tag{8}$$

Build a GE Model

- (1) Zero-profit equation for welfare.
- (2) Zero-profit equation for G and S.
- (3) Market-clearing for aggregate demand.
- (4) Market-clearing for G and S.
- (5) Market-clearing for labor.
- (6) Income definition.

Exogenous Environment

These are the **data** that we need to collect and **fixed parameters** that we need to estimate or sometimes get elsewhere.

- (1) Labor endowment (L): 100.
- (2) Subsistence consumption (Γ_i): $\bar{G} = 40$ and $\bar{S} = 0$.
- (3) Fixed parameters: $\alpha_G = 1/6$ and $\alpha_S = 1 - \alpha_G$.
- (4) Define: ω is the labor wage per unit of goods.
- (5) Define: M as income.
- (6) Define: PU as the price index. It can be derived by solving the marginal income of utility in the optimization problem, and then substitute the expression into the LES Hicksian expenditure function.

Corresponding Mixed Complementarity Conditions

- (1)
$$\prod_i p_i \alpha_i \geq PU \quad \perp \quad u \geq 0.$$
- (2)
$$\alpha_i \omega_i \geq p_i \quad \perp \quad X_i \geq 0.$$
- (3)
$$PU = (M - \sum_i \Gamma_i)/U.$$
- (4)
$$X_i \geq \Gamma_i + \alpha_i U(PU)/p_i \quad \perp \quad p_i \geq 0.$$
- (5)
$$L \geq \sum_i \alpha_i X_i \quad \perp \quad \omega \geq 0.$$
- (6)
$$M = \omega L.$$