ESTIMATION OF AN IMPLICIT ADDITIVE INDIRECT DEMAND SYSTEM

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Preference structures in applied general equilibrium models are often limited to constant-elasticity-of-substitution (CES) forms due to the desire for global regularity. Hanoch (1975) presents indirect, implicit additive relationships as a generalization of CES, providing more flexible and globally regular demand relationships. These preferences decouple substitution and income effects beyond homotheticity relaxation and exhibit greater flexibility than their direct dual. However, the estimation of these models as demand systems has proven to be challenging, with most published work in this area focusing on estimation approaches that involve approximations or cannot fully identify parameter values in the preference relationships. Our approach is direct, avoids approximations, and allows parameter identification. Using the unpublished World Bank (International Comparison Program) database, we estimate the constant difference of elasticity or CDE directly in a maximum likelihood framework. We show that the global regularity conditions stated in Hanoch (1975) can be slightly relaxed, and that the relaxed parametric conditions facilitate estimation. We introduce a normalization scheme that is beneficial for the scaling of the parameter values and which appears to have little impact on the economic performance of the estimated system. Our methodology bears a conceptual resemblance to Berry (1994), yet sets itself apart by calculating utility levels via constrained optimization, all while maintaining a transparent and tractable numerical procedure for estimating general demand models.

KEYWORDS: Demand system, implicit additivity, parameter estimation, constant difference of elasticity.

1. INTRODUCTION

Preference structures in applied general equilibrium models are often limited to constant-elasticity-of-substitution (CES) forms due to the desire for global regularity.[1] Hanoch (1975) uses implicit additive relationships—a generalization of the CES—to obtain more flexible demand relationships that are globally regular. These models are both parsimonious, as the number of parameters is approximately proportional to the number of goods, and flexible, allowing goods to be substitutes/complements and normal/inferior (as described in Section 2). One of the key advantages of these implicit models is that their substitution matrices are less constrained than standard CES models, where Allen-Uzawa partial elasticities of substitution depend on income elasticities, and a ratio or additive relationship always exists between them (Houthakker (1960), Hanoch (1975)). The implicit additive models relax these restrictions on substitution effects, which can be expressed without reference to income effects, and thus effectively avoid some of the constraints on their relationships due to *Pigou's Law* (Pigou (1910), Deaton (1974),

[1]A noteworthy contribution to the flexible functional forms in applied general equilibrium literature is Perroni and Rutherford (1995). They introduce a class of globally regular, locally flexible, nested CES functions.

Barten (1977)). Moreover, implicit additive models can even accommodate cases with no substitution among goods, a scenario that is possible in the data (Hanoch, 1975).

Recent studies have brought a newfound focus on demand models which integrate these more flexible implicit preference relationships into the literature of macroeconomics, international trade, and spatial models. For example, Comin, Lashkari, and Mestieri (2021) estimate an implicit additive *direct* model by leveraging both household and macro-level data in a general equilibrium context featuring structural change and economic growth. Yang (2021) develops a general equilibrium gravity model of factor trade based on the implicit additive *indirect* model, structurally identifying model parameters using the standard gravity equation and population data; despite their flexibility, however, there has been little to no work on econometric identification for the implicit additive demand models in the past literature.

The majority of literature addressing the identification of implicit additive demand models focuses on implicitly additive direct systems. For instance, Rimmer and Powell (1996) develop an Implicitly Directly Additive Demand System (AIDADS) and estimate its parameters using a recursive approach based on ordered real income per head data points, following the suggestion of McLaren (1991). Their model can be considered a generalization of the Linear Expenditure System and has been empirically investigated by Cranfield et al. (2002). Preckel, Cranfield, and Hertel (2010) propose a more generalized implicit additive direct model, which has been estimated by Gouel and Guimbard (2019) to examine the structure of global food demand.

Notably, most demand literature overlooks the advantages of implicitly additive indirect models over their direct dual. We emphasize that one significant flexibility of the indirect forms, as discussed in Hanoch (1975), is the possibility of goods being complements even under global regularity restrictions. While such a feature exists in direct forms, it only holds locally and is incompatible with global regularity restrictions, as implicit additive direct models that are valid globally restrict all pairs of goods to be substitutes (Hanoch, 1975). This constraint on preferences for goods is generally unrealistic. Therefore, indirect, implicit additive models emerge as more suitable candidates for representing consumer preferences.

The direct estimation of implicit additive indirect models as demand systems has proven to be extremely challenging, with most published work in this area focusing on approximation-based approaches (Pudney (1981), Hertel et al. (1991), Liu et al. (1998), Chen (2017)). One empirical reason for these approximations is that utilities remain unobservable to econometricians. Consequently, these models have generally been estimated and used as production functions where the analogue to utilities is observable production output (Hawkins (1977), Merrilees (1982), Dar and Dasgupta (1985), Surry (1993), Hashimoto and Heath (1975)). As for the work involving the estimation of these models as demand systems, the demand literature suggests that any reduced-form approach requires double log-differencing to eliminate utilities, ultimately facing parameter identification issues. Other empirical work on these models uses entropy approaches and calculates demand parameters from income and price elasticities of other estimable demand systems, such as the Linear Expenditure System proposed by Stone (1954) and the AIDADS family (Rimmer and Powell (1996), Preckel, Cranfield, and Hertel (2010)).

We contribute to the literature by demonstrating the direct estimation of the indirect, implicit additive model as a demand system for the first time. We use an unpublished World Bank database to develop a dataset with expenditure and price data suitable for estimating implicit preferences. We then estimate an implicit additive indirect demand relationship, the constant difference of elasticity (CDE), directly in a maximum likelihood framework. In doing so, we sharpen the global regularity conditions stated in Hanoch (1975), resulting in slightly relaxed conditions that facilitate estimation. Furthermore, we introduce a normalization scheme that enables parameter identification, aids in scaling the parameter values, and appears to have minimal impact on the economic performance of the estimated system.

Our approach shares a conceptual connection with the groundbreaking work of Berry (1994), who introduces an important method for estimating discrete-choice models in the industrial organization literature by inverting market shares and computing unobservable utility levels. In contrast, we employ observed "share information" in our estimation system, leveraging constrained optimization as illustrated in Su and Judd (2012). This allows us to calculate utility levels explicitly without capturing any deviations from the mean demand shifters as "errors". Despite the apparent dissimilarities between our paper and Berry's work, such as Berry's estimation of a discrete-choice demand model using product market shares, we wish to highlight the shared empirical intuition. In both approaches, utility is made explicit by utilizing the shares. Indeed, our methodology potentially lends itself to extension through the incorporation of instrumental variables as in Berry (1994), yet such an extension is beyond the scope of this paper. Our approach can likely be extended to estimate implicit production relationships, where, for example, an aggregate sectoral output in a nested structure, which is unobserved by the econometrician (Berry, Searchinger, and Yang, 2024), or to apply it in contexts where it relates other sources of exogenous variation to unobservables in the model.

2. THEORY OF THE IMPLICIT CDE FUNCTIONAL FORM

The demand model examined in this paper is an implicit and indirect relationship that relates utility, prices and total expenditure as follows:

$$G\left(\frac{\mathbf{p}}{w}, u\right) = \sum_k \beta_k u^{e_k(1-\alpha_k)} \left(\frac{p_k}{w}\right)^{1-\alpha_k} \equiv 1, \tag{2.1}$$

with $\log[u^{e_k}(p_k/w)]$ replacing $u^{e_k(1-\alpha_k)}(p_k/w)^{1-\alpha_k}$ in the limiting case where α_k approaches unity and where the subscript $k \in \{1, ..., N\}$ indexes commodities; with vector $\mathbf{p} = \{p_k\}_{k=1}^N$ denotes commodity prices; u denotes per capita utility, and w denotes the per capita total expenditure. The model parameters to be estimated, β's, e's, and α's are distribution, expansion and substitution parameters, respectively (Hanoch, 1975). In addition, the levels of per capita utility for each country are estimated. While it is unusual to estimate unobservable utility in demand studies, when estimation of an underlying explicit demand system derived from a utility maximization problem (e.g., in the CES case) is undertaken, the estimation produces everything needed to calculate utility up to a strictly increasing transformation. The difference lies in the fact that with an explicit functional form, there is no need to estimate utility, whereas with an implicit functional form utility must be explicitly estimated to fully identify the model parameters. As to the data, the prices are compiled from the ICP database for 2017; w is obtained from the ICP database; and p_k/w may be interpreted as the unit-cost price or the normalized price of commodity k; and quantities, which do not appear in equation (2.1), but are nonetheless important. The stated parametric restrictions for the demand function to be globally valid (monotonic and quasi-concave) are that, at all $\mathbf{p}/w \gg 0$ (i.e., unit-cost prices all strictly positive), (i) $\beta_k, e_k > 0 \ \forall \ k \in N$, and (ii) either $\alpha_k > 1$ or $0 \leq \alpha_k \leq 1 \ \forall \ k \in N$. (The weak inequalities in the second set of conditions in (ii) are justified in Section 4.3).

The model is categorized as an implicit (rather than an explicit) function because the relationahip defined by equation (2.1) cannot in general be algebraically solved for utility as an explicit function of exogenous variables and parameters. The model is indirect because its indifference curves, which illustrate demand patterns, are expressed in its unit-cost prices instead of quantities. The model is closely related to other standard demand models. For example, it is easy to show that, if we set $e_k = e = 1 \ \forall \ k \in N$ and $\alpha_k = \alpha \ \forall \ k \in N$, then equation (2.1) collapses to the standard indirect CES model.

Using Roy's Identity, the derived demand correspondence is

$$q_k\left(\frac{\mathbf{p}}{w}, u\right) = \frac{w}{p_k}\frac{\theta_k}{\sum_j \theta_j} = \frac{w}{p_k}\Lambda_k, \qquad (2.2)$$

where θ_k is an auxiliary variable such that

$$\theta_k = \beta_k u^{e_k(1-\alpha_k)}(1-\alpha_k)\left(\frac{p_k}{w}\right)^{1-\alpha_k}, \qquad (2.3)$$

and Λ_k is the expenditure shares of goods k as a function of θ_k, which equals

$$\Lambda_k\left(\frac{\mathbf{p}}{w}, u\right) = \frac{\theta_k}{\sum_j \theta_j}. \qquad (2.4)$$

Incorporating our econometric method, outlined below in Section 4, and combining it with Equation (2.3), we can interpret Equation (2.4) as a vector-valued equation, where $\mathbf{\Lambda} = \mathbf{G}(\mathbf{u})$ and $\mathbf{u}$ represents the *utility level* allowing an exact fit of the model without the presence of the parametric distribution of *unobserved factors*. Equation (2.4), which represents an expenditure share function of utilities, is not fundamentally different from the logit model or Bresnahan (1987)'s vertical differentiation model discussed in the two elementary special cases in Berry (1994). While our model is implicit and indirect, so too is the log-differencing approach introduced in Section 3.1, which linearizes the equations in a fashion similar to Berry (1994). In line with Berry (1994) and Berry and Haile (2021), who relate the shares or other valid sources of exogenous variation to unobservables, we exploit the relationship (2.4) between the shares and unobservable utilities, model parameters, and the normalized prices.

The Allen-Uzawa elasticities of substitution σ_{km} are given by

$$\sigma_{km} = \alpha_k + \alpha_m - \sum_j \Lambda_j \alpha_j - \frac{\Delta_{km}\alpha_k}{\Lambda_k}, \qquad (2.5)$$

where Δ_{km} is the Kronecker delta (equaling 1 if $k = m$; 0 if otherwise). Since σ_{km} is derived as a function of the share-weighted sum of expansion parameters, it can be negative (and thus complementary goods k and m may exist for $N \geq 3$) if the latter is large, or alternatively, if the expenditure share of goods k, which can possibly have a large substitution elasticity, is small.

The income elasticities η_k are given by

$$\eta_k = \frac{e_k(1-\alpha_k) + \sum_j \Lambda_j e_j \alpha_j}{\sum_j \Lambda_j e_j} + \alpha_k - \sum_j \Lambda_j \alpha_j. \qquad (2.6)$$

It can be readily observed from equation (2.6) that goods are allowed to be inferior rather than normal, i.e., η_k can be negative. Again, this can happen if $\sum_k \Lambda_k \alpha_k$ is sufficiently large.

A key advantage of the implicit model in question, as shown in (2.5), is its greater flexibility in comparison to explicit models, as it imposes fewer restrictions on the substitution matrix. As demonstrated by Houthakker (1960) and Hanoch (1975), explicit models exhibit

two types of tight linkages between substitution and income elasticities. One is that the Allen-Uzawa substitution elasticities can always be derived as functions of the income elasticities, i.e., $\sigma_{km} = \eta_k \eta_m (\sum_j \alpha_j \Lambda_j)$ for the explicitly direct models, such as the CES model, or $\sigma_{km} = \eta_k + \eta_m + (\sum_j \alpha_j \Lambda_j - 2)$ for the explicitly indirect models. The other linkage is that there is always a ratio or additive relationship between the substitution and income elasticities, i.e., $\eta_k / \eta_m = \sigma_{kj} / \sigma_{mj}$ for the direct case, or $\eta_k - \eta_m = \sigma_{kj} - \sigma_{mj}$ for the indirect case (Hanoch (1975), Yang (2021)).[2] In contrast, the implicit model allows for a more flexible representation of consumer behavior (or input demand as a production function), without relying on rigid linkages between substitution and income (or expansion) effects. This offers a more adaptable approach to understanding economic responses to shocks, providing better economic justification, especially in cases where substitution is not clearly observed in the data.

3. IDENTIFICATION ISSUES

There are two major issues with identification related to the estimation of implicitly additive preferences. The first issue applies to reduced-form approaches, where the transformations to eliminate utility from the demand system also eliminate the possibility of identifying all parameters in the system (in addition to endogeneity problems). The second issue is due to unobserved "normalizing constant" which is fundamental to the CDE functional form (and even in more restricted cases in the Bergson family). We demonstrate that this can be resolved by imposing normalizations as indicated in Section 4.1.

3.1. *Reduced-Form Approaches*

The previous demand literature suggests that any reduced-form approach (i.e., ordinary least squares regression) for estimating these implicit indirect models as demand systems would require double log-differencing to eliminate utilities, but this would ultimately result in identification problems. To see this, we take the natural log of both sides in equation (2.2) of the demand correspondence (with $\xi_k = p_k / w$):

$$
\ln q_k = \ln \left[\beta_k u^{e_k(1-\alpha_k)} (1-\alpha_k) \xi_k^{-\alpha_k} \right] - \ln \left[\sum_j \theta_j \right]
$$
$$
= \ln[\beta_k(1-\alpha_k)] + e_k(1-\alpha_k) \ln u - \alpha_k \ln \xi_k - \ln \left[\sum_j \theta_j \right].
$$
(3.1)

Eliminating the last term in (3.1) by using the logarithmic ratio:

$$
\ln \frac{q_k}{q_1} = \ln \frac{\beta_k(1-\alpha_k)}{\beta_1(1-\alpha_1)} + [e_k(1-\alpha_k) - e_1(1-\alpha_1)] \ln u - \alpha_k \ln \xi_k + \alpha_1 \ln \xi_1
$$
$$
= A_k + Z_k \ln u - \alpha_k \ln \xi_k + \alpha_1 \ln \xi_1 + \epsilon_k,
$$
(3.2)

where $A_k = \ln\{\beta_k(1-\alpha_k)/[\beta_1(1-\alpha_1)]\}$ and $Z_k = e_k(1-\alpha_k) - e_1(1-\alpha_1)$; q_1 is the chosen good for normalization; $\epsilon_k \forall k \in N-1$ is assumed to be the random error, which is independent of ξ_k, and has mean zero and constant variance.

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[2]These dependencies are true for all explicitly direct or indirect additivities (see Hanoch (1975)). In the literature of the 1960s, these relationships are sometimes referred to as relationships between cross-price derivatives and income derivatives, or substitution effects and Engel derivatives (Houthakker (1960), Powell (1966)).

Note that the estimation equation (3.2) might be more suitable for the function used as a production function, such as in Surry (1993) where u is the observable level of output (rather than utility). Because the utility levels are unknown in the demand context, we cannot directly estimate (3.2) by performing a regression of the logarithmic ratio of quantities on the explanatory variables on the right-hand side. In order to estimate (3.2) as a demand function, we shall first eliminate the unobservable u. One way to do this is to choose q_2 as a second good for normalization:

$$
\begin{aligned}
\ln \frac{q_2}{q_1} &= \ln \frac{\beta_2(1-\alpha_2)}{\beta_1(1-\alpha_1)} + [e_2(1-\alpha_2) - e_1(1-\alpha_1)] \ln u - \alpha_2 \ln \xi_2 + \alpha_1 \ln \xi_1 \\
&= A_2 + Z_2 \ln u - \alpha_2 \ln \xi_2 + \alpha_1 \ln \xi_1 + \epsilon_2,
\end{aligned}
\tag{3.3}
$$

where $A_2 = \ln\{\beta_2(1-\alpha_2)/[\beta_1(1-\alpha_1)]\}$ and $Z_2 = e_2(1-\alpha_2) - e_1(1-\alpha_1)$.

Isolating (3.3) so that only u is on the left-hand side:

$$
\ln u = \frac{-A_2 + \alpha_2 \ln \xi_2 - \alpha_1 \ln \xi_1 + \ln \dfrac{q_2}{q_1} - \epsilon_2}{Z_2}.
\tag{3.4}
$$

Now substituting (3.4) into (3.2) to eliminate u without loss of generality in terms of the functional form, with $R_k = Z_k/Z_2$:

$$
\begin{aligned}
\ln \frac{q_k}{q_1} &= A_k + \frac{Z_k}{Z_2}\left(-A_2 + \alpha_2 \ln \xi_2 - \alpha_1 \ln \xi_1 + \ln \frac{q_2}{q_1} - \epsilon_2\right) \\
&\quad - \alpha_k \ln \xi_k + \alpha_1 \ln \xi_1 + \epsilon_k \\
&= A_k - R_k A_2 + R_k \alpha_2 \ln \xi_2 - R_k \alpha_1 \ln \xi_1 + R_k \ln \frac{q_2}{q_1} \\
&\quad - \alpha_k \ln \xi_k + \alpha_1 \ln \xi_1 - R_k \epsilon_2 + \epsilon_i.
\end{aligned}
\tag{3.5}
$$

By rearranging equation (3.5), we now obtain an estimation equation as follows:[3]

$$
\ln \frac{q_k}{q_1} = S_k - \alpha_k \ln \xi_k + R_k \alpha_2 \ln \xi_2 + (1-R_k)\alpha_1 \ln \xi_1 + R_k \ln \frac{q_2}{q_1} + \varphi_k,
\tag{3.6}
$$

where $\varphi_k = \epsilon_k - R_k \epsilon_2$ is the error term $\forall\, k \in N-2$, satisfying standard assumptions as for ϵ_k.

Note that the regression estimation to equation (3.6) immediately yields the intercept S_k, and coefficients α_k, $R_k \alpha_2$, $(1-R_k)\alpha_1$ and R_k, which automatically yields α_1 and α_2. Given R_k, the relationship between e_k and e_2 can be identified. Therefore, the system can be completely solved if β_2 and e_2 are pinned down, given S_k, which cannot be accomplished without further identities. With this approach, it is clear to see that only $N-2$ substitution parameters α_k

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[3]In practice, empirical econometric work often concerns the simultaneity bias issues (i.e., income which equates total expenditure is jointly determined by the quantity demanded for goods k) and the fact that the arbitrary choice of normalized goods can lead to multiple parameter estimates at length N (Surry, 1993). In this case, the estimation equation can be modified to: $\ln \Lambda_k/\Lambda_1 = S_k + (1-\alpha_k)\ln \xi_k + R_k \alpha_2 \ln \xi_2 + [(1-R_k)\alpha_1 - 1]\ln \xi_1 + R_k \ln q_2/q_1 + \varphi_k$ (by letting $\Lambda_k = q_k(p_k/w) = q_k \xi_k$ so that $\ln q_k/q_1 = \ln \xi_k + \ln \xi_1$). This, however, raises isomorphic identification issues as from equation (3.6).

can be estimated, but we cannot solve for expansion parameters e_k and distribution parameters β_k. Thus, given the model specification, the reduced-form estimation framework presents challenges for parameter identification.

Furthermore, it is important to note that ξ_k (unit-cost price) is very likely correlated with φ_k, while finding appropriate instruments is difficult given the limited data in our cross-sectional setting (Berry (1994), Berry, Levinsohn, and Pakes (1995)). In this case, endogeneity can lead to biased and inconsistent estimates. The method we introduce in Section 4.3, though requiring increased computational effort, offers a numerical procedure based on constrained optimization. This enables the calculation of u satisfying the defining relationship (2.1), while achieving the identification of other model parameters (i.e., α_k, β_k and e_k). Our proof of regularity condition in Section 4.3 shows there exists a unique solution to (2.1) up to a strictly increasing transformation of u as is introduced in Section 4.1.

3.2. *Excess Degrees of Freedom*

Since u cannot be observed, the preference defining relationship in equation (2.1) has multiple sets of parameter values that will satisfy the relationship equally well. To see this, note that since any strictly increasing transformation of utility will not alter the ordering of preferences for alternative consumption bundles, such transformations will have no impact on the quality of the estimated relationship. For example, consider the transformation $u = \rho \upsilon^\delta$ where $\upsilon > 0$, which will be strictly increasing if $\rho > 0$ and $\delta > 0$. In this case, υ will be as good as u for explaining the data. Substituting the transformed u into equation (2.1):

$$
\begin{aligned}
1 \equiv G\Big(\frac{\mathbf{p}}{w}, u\Big) &= \sum_k \beta_k u^{e_k(1-\alpha_k)} \Big(\frac{p_k}{w}\Big)^{1-\alpha_k} \\
&= \sum_k \beta_k \big[\rho \upsilon^\delta\big]^{e_k(1-\alpha_k)} \Big(\frac{p_k}{w}\Big)^{1-\alpha_k} \\
&= \sum_k \beta_k \rho^{e_k(1-\alpha_k)} \upsilon^{\delta e_k(1-\alpha_k)} \Big(\frac{p_k}{w}\Big)^{1-\alpha_k} \\
&= \sum_k \widetilde{\beta}_k \upsilon^{\widetilde{e}_k(1-\alpha_k)} \Big(\frac{p_k}{w}\Big)^{1-\alpha_k},
\end{aligned}
\tag{3.7}
$$

where $\widetilde{\beta}_k = \beta_k \rho^{e_k(1-\alpha_k)}$, and $\widetilde{e}_k = \delta e_k$. Since ρ and δ were any strictly positive constants, there is a continuum of values for the $\widetilde{\beta}_k$ and $\widetilde{e}_k$ that exactly satisfy equation (2.1) given that this defining relationship is satisfied for β_k and e_k. Thus, these parameters are not fully identified. For this reason, we introduce normalizations for these two sets of parameters, choosing our normalizations in such a way that the estimated parameter values are "well-scaled" as we will describe in Section 4.1.

4. ECONOMETRIC PROCEDURE

We follow the maximum likelihood estimation procedure for implicit additive direct demand systems used in Cranfield et al. (2002), Preckel, Cranfield, and Hertel (2010), Gouel and Guimbard (2019), and Yang, Gouel, and Hertel (2018). In this framework, we estimate the implicit indirect relationship using constrained maximum likelihood subject to a set of constraint equations and parametric restrictions that enforce global regularity. The concentrated log-likelihood $\log \mathcal{L}_c$, as discussed in Greene (2012), is given by:

$$\log \mathcal{L}_c = -\frac{I}{2}[J(1 + \log 2\pi) + \log |\mathbf{R'R}|] \quad (\text{with } \hat{\beta}_{\mathrm{ML}} = \mathrm{Min}_\beta \frac{1}{2} \ln |\mathbf{R'R}|), \qquad (4.1)$$

where I and J are the numbers of countries (or regions) and commodities (or services) in our data set, respectively; $|\mathbf{R'R}|$ is the determinate of the cross-commodity (or service) error covariance matrix. The disturbances (expressed in d's) are the residuals between actual and fitted budget shares:

$$d_{ik} = \Lambda_{ik} - \widehat{\Lambda}_{ik}, \qquad (4.2)$$

where i and k index countries and goods, respectively; $\widehat{\Lambda}_{ik}$ are the fitted expenditure shares, and the components of $\mathbf{R}$, r_{nm}, are constrained by:

$$\sum_n r_{nk} r_{nm} = \sum_i \frac{d_{ik} d_{im}}{I}, \qquad (4.3)$$

along with $r_{km} = 0$ for all $m > k$, making $\mathbf{R} = [r_{km}]_{k,m=1,\dots,N}$ an upper triangular Cholesky factorization of the error covariance matrix. The advantage of working with this factorization of the cross-equation error covariance matrix is that evaluation of the determinate of the covariance matrix is simple—it is the square of the product of the diagonal elements of $\mathbf{R}$.

4.1. *Identification and Normalization Strategy*

As shown above, the system and the model parameters are not fully identified without removing excess degrees of freedom from the parameter space. In realated work, Comin, Lashkari, and Mestieri (2021) study a less general implicit direct non-homothetic CES demand system:

$$G\left(\frac{\mathbf{p}}{w}, u\right) = \sum_k \beta_k u^{e_k(1-\alpha)} \left(\frac{p_k}{w}\right)^{1-\alpha} = 1, \qquad (4.4)$$

which can be obtained by restricting $\alpha_k = \alpha \; \forall \; k$ in equation (2.1).[4] The authors developed an approach to estimation that influences ours and that employs normalizations of parameter space that are similar in spirit to ours.

Comin, Lashkari, and Mestieri (2021) observe that the expressions for the own price and income elasticities of demand are invariant to a multiplicative scaling of the parameters equivalent to our β_k and e_k for the direct non-homothetic CES case. In the interest of parameter identification, they remove a degree of freedom for each of these sets of parameters by normalizing these parameters to unity for one good (e.g., $k = 1$). We show in Section 3.2 the same invariance to rescaling of these parameters in the indirect case. However, we choose a slightly different normalization scheme by setting $\sum_k \beta_k \equiv 1$ and $\sum_k e_k \equiv N$, again removing one degree of freedom for each of these sets of parameters. Our normalization of the β_k combined with the form of our implicit preference defining relationship (2.1) will tend to improve the scaling of the terms $u^{e_k(1-\alpha_k)}(p_k/w)^{1-\alpha_k}$, which we have observed tends to improve the scaling of the parameters α_k. Similarly, the choice to normalize the sum of the exponents e_k to

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[4]The system specified by Comin, Lashkari, and Mestieri (2021) is $\sum_k (\Omega_k C^{\epsilon_k})^{1/\sigma} C_k^{(\sigma-1)/\sigma} \equiv 1$, where C_k denotes demand, C is an aggregator index, Ω_k, σ, and ϵ_k are parameters. This can be viewed as a (restricted) direct form of the demand relationship discussed in 4.4.

be equal to the number of goods (and thus each e_k equals to one on average across the goods) tends to improve the scaling of the u levels. Our motivation, as well as that of Comin, Lashkari, and Mestieri (2021), is to obtain parameter identification for our demand system. While our normalizations are mathematically equivalent, our strategy of normalizing sums of these parameters rather than individual ones avoids the possibility of making an unfortunate choice for the parameters that are set to one, which may serve to make the other parameters either large or small numbers. Given that we solve our estimation problems numerically using general non-linear programming software, attention to scaling can improve our likelihood of success in estimation.

4.2. *Constrained Optimization*

Our constrained optimization problem can be viewed as a subset of the general approach proposed by Su and Judd (2012), who illustrated a Mathematical Programming with Equilibrium Constraints (MPEC) approach for estimating structural econometric models. In contrast to their approach, our method is more in line with estimating structural *partial equilibrium* models, which focus solely on the demand side. This is similar to an estimation of Anderson and Van Wincoop (2003)'s 'Armington' model using an MPEC approach (which is equivalent to a non-linear programming like ours), as demonstrated by Balistreri and Hillberry (2007).

The constrained optimization problem is to maximize equation (4.1), subject to the constraints (4.2), (4.3) along with $r_{km} = 0 \ \forall \ m > k$, the implicit indirect additivity relationship (4.4), the normalization equations $\sum_k \beta_k = 1$ and $\sum_k e_k = N$, the auxiliary identities (2.2)–(2.4), as well as the *redefined* parametric restrictions of the demand system.

As we show in Section 4.3, the strictly positive lower bounds on α_k's are not essential, although in the direct case corner solutions may result from the utility maximization problem. Since, the quasi-concavity properties from the direct case are inherited by the indirect case, the demand system is well-behaved with these relaxed lower bounds for α_k. Consequently, we impose weak inequalities for the lower bounds on the α's, and our parametric constraints consist of (i) $\beta_k, e_k > 0 \ \forall \ k \in N$, and (ii) $0 \le \alpha_k \le 1 \ \forall \ k \in N$ (refer to Section 4.3). This choice disregards the situation where $\alpha_k > 1$, which we determined to be irrelevant for our dataset because the resulting elasticity estimates lacked credibility. (Our formulation applies equally well to the case where $\alpha_k > 1 \ \forall \ k \in N$.) Following Hanoch (1975), we interpret $(p_k/w)^{1-\alpha_k}$ as $\ln(p_k/w)$ when $\alpha_k = 1$; however, in our empirical analysis it emerged that $\alpha_k < 1 \ \forall \ k \in N$.

4.3. *Proof of Global Regularity*

In reviewing the published regularity conditions in Hanoch (1975) we noticed what appear to be some minor discrepancies in the conditions related to the boundaries for the parameter α_k.[5] In this section, we readdress the proof of global regularity with an eye to whether the extreme values $\alpha_k = 0$ for some k should be included.

The CDE preference relationship is said to be globally regular if the relationship in equation (2.1) is monotone in both u and p_k/w and satisfies a quasi-concavity property. To show this, it is useful to begin from the direct form of the CDE, the CRES. We demonstrate that under a mildly

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[5]In Hanoch (1975) (p. 403), with different subscript i, the stated parametric restrictions for $d_i (= 1 - a_i)$ in the CRES Model (i.e., equation (2.16)) is that either $0 < d_i < 1$ or $d_i \le 0$, $\forall \ i$, and for a_i is that either $a_i > 1$ or $0 < a_i \le 1 \ \forall \ i$; then on p. 411, the stated conditions for $b_i (= 1 - \alpha_i)$ in the CDE Model (i.e., equation (3.15)) is that either $0 < b_i < 1$ or $b_i \le 0 \ \forall \ i$, and for α_i is that either $\alpha_i \ge 1$ or $0 < \alpha_i < 1 \ \forall \ i$. Because d_i's and b_i's as well as a_i's and α_i's are used interchangeably, the same regularity conditions for d_i and b_i should imply the same restrictions for a_i and α_i which, however, are not what we find in Hanoch (1975).

relaxed set of parametric restrictions relative to Hanoch (1971, 1975), that we obtain global regularity for the CRES. Hanoch (1975) argues that regularity of the direct CRES relationship yields regularity of the indirect CDE relationship due to symmetry between $f(\mathbf{x})$ and its *indirect reciprocal* $g(\mathbf{p}/w)$ in the CDE.

Following Hanoch (1975), we begin by setting up the utility maximization problem for the CRES subject to the defining constraint for the CRES relationship. This relationship has near-identical form to the CDE as follows:

$$\mathscr{H}(\mathbf{x}, u) = \sum_k \beta_k u^{-e_k(1-\alpha_k)}(x_k)^{1-\alpha_k} \equiv 1, \tag{4.5}$$

with $\log(x_i/u^{-e_k})$ replacing $u^{-e_k(1-\alpha_k)}(x_k)^{1-\alpha_k}$ in the limiting case where α_k approaches unity and where $\mathbf{x}$ denotes the levels of inputs. The stated regularity conditions are: (i) $\beta_k, e_k > 0 \ \forall \ k \in N$, and (ii) either $\alpha_k > 1$ or $0 < \alpha_k \le 1 \ \forall \ k \in N$.

THEOREM 4.1: *The latter set of global regularity conditions in equation (4.5), i.e., condition (ii): either $\alpha_k > 1$ or $0 < \alpha_k \le 1 \ \forall \ k \in N$, can be relaxed to either $\alpha_k > 1 \ \forall \ k \in N$, or $0 \le \alpha_k \le 1 \ \forall \ k \in N$.*

PROOF: The case where $\alpha_k > 1$ for all k is covered in the original work by Hanoch (1975) (Section 2.4, p.403) and is not addressed here.[6] Similarly, the interpretation of $\mathscr{H}(x, u)$ when some $\alpha_k = 1$ can be treated as in Hanoch (1975), and does not affect the qualitative properties of $\mathscr{H}(x, u)$. Hence, it is sufficient to focus on the case where $0 \le \alpha_k < 1$ to demonstrate regularity with the relaxed lower boundary condition on α_k. Also note that the CRES is linear in x_k for the case where $\alpha_k = 0$ for all k. This case is not of central interest here, and so we assume that $\alpha_k > 0$ for at least one k.

The utility maximization problem that motivates the demand system is:

$$\begin{aligned} & \underset{x_k \ge 0 \ \forall k \in N}{\text{maximize}} \quad u \\ & \text{subject to:} \quad \mathscr{H}(x, u) = \sum_k \beta_k u^{-e_k(1-\alpha_k)}(x_k)^{1-\alpha_k} = 1 \\ & \hspace{4.5cm} \sum_k c_k x_k \le I, \end{aligned} \tag{4.6}$$

for fixed I, which has the dual expenditure minimization problem:

$$\begin{aligned} & \underset{x_k \ge 0 \ \forall k \in N}{\text{minimize}} \quad \sum_k c_k x_k \\ & \text{subject to:} \quad \sum_k \beta_k u^{-e_k(1-\alpha_k)}(x_k)^{1-\alpha_k} = 1, \end{aligned} \tag{4.7}$$

for fixed $u > 0$. Note that each term in the sum in the constraint of the latter problem is strictly increasing in x_k (for $0 \le \alpha_k < 1$, $\beta_k > 0$, and $e_k > 0$), decreasing in u, and that $\mathscr{H}(x, u)$ is strictly decreasing in u. The importance of these features are two-fold. First, the fact that each

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[6] The concept of *constant ratios of elasticities of substitution* or CRES was discussed in Mukerji (1963), who explore a generalized version of the Solow, Minhas, Arrow, and Chenery function (Arrow et al., 1961).

term in strictly increasing in x_k and that $\mathscr{H}(x,u)$ is strictly decreasing in u implies that u is strictly increasing in each x_k. This demonstrates a strict monotonic relationship between u and each x_k. Second, the fact that $\mathscr{H}(x,u)$ is strictly decreasing in u combined with the fact that each term in $\mathscr{H}(x,u)$ increases without bound as u approaches zero and approaches zero as u increases without bound, implies that the solution to $\mathscr{H}(x,u)=1$ will be unique. Thus, the implicit relationship defined by $\mathscr{H}(x,u)=1$ defines a single value for u for any given x.

The first-order condition w.r.t. x_k is

$$c_k - \lambda\beta_k u^{-e_k(1-\alpha_k)}(1-\alpha_k)x_k^{-\alpha_k} \geq 0. \tag{4.8}$$

For any k such that $\alpha_k > 0$ this condition can only be satisfied with $x_k > 0$ and $\lambda > 0$, implying that the equality relationship for the constraint in the expenditure minimization problem (4.7) can be replaced with "$\geq$" without loss of generality.

The Hessian of the constraint in the expenditure minimization problem is diagonal:

$$\nabla_x^2 \mathscr{H}(x,u) = \text{Diag}\left\{\beta_k u^{-e_k(1-\alpha_k)}(1-\alpha_k)(-\alpha_k)x_k^{-\alpha_k-1}\right\}, \tag{4.9}$$

and each diagonal element is either strictly negative if $0 < \alpha_k < 1$ or zero if $\alpha_k = 0$. Thus, $\mathscr{H}(x,u)$ is a concave function and $\{x|\mathscr{H}(x,u) \geq 1\}$ is a convex set. Finally, because $\mathscr{H}(x,u)$ is strictly increasing in u, $\{x|\mathscr{H}(x,u) \geq 1 \text{ and } u \geq a\}$ is also a convex set, and so $\mathscr{H}(x,u)=1$ defines a quasi-concave relationship between u and x.   *Q.E.D.*

In Appendix B, we provide solutions to the expenditure minimization problem under relaxed regularity conditions. It is lengthy, so we have omitted it here. The proof leads us to our relaxed set of parametric conditions for the CRES: $\beta_k, e_k > 0 \; \forall \; k \in N$, and either $\alpha_k > 1 \; \forall \; k \in N$, or $0 \leq \alpha_k \leq 1 \; \forall \; k \in N$. Again, following Hanoch (1975), due to *complete formal symmetry* between the direct and indirect cases, global regularity of the CRES in $\mathbf{x} \geq 0$ implies global regularity of the CDE in $\mathbf{p}/w \gg 0$. Note that the corner solutions for the CRES do not similarly imply that the CDE will generate corner solutions when some of our parametric restrictions on α_k are binding. This is because the envelope conditions used to reclaim the demand quantities involve not only derivatives with respect to the numerator in p_k/w, but also in the denominator, which appears in other terms in our defining equation.

5. DATA

The data set used for our analysis includes information on commodity and service expenditures, prices, exchange rates, and regional population, all obtained from the World Bank's ICP database for the reference year 2017. The World Bank has calculated Purchasing Power Parities (PPPs) for 109 different commodities and services in 216 regions at the level of *basic headings*, which provide detailed information for specific types of goods and services.[7] However, due to missing expenditure or price information, our estimation sample includes only 178 regions.

The data set includes a wide range of commodities and services, spanning categories such as food, housing, transportation, communication, and many others. By using basic headings, we can obtain more accurate estimates of PPPs for each of these specific types of goods and services. The unpublished ICP data set provides us with the opportunity to conduct a comprehensive analysis of the underlying data and gain a deeper understanding of the price variations and expenditures across these more specific categories before aggregating at a higher level, which is crucial for our estimation process.

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[7]In the ICP data of the World Bank, 'basic heading' is the most detailed or granular level at which participating economies report their expenditure values.

5.1. *Implicit Additivity for Analyzing Goods and Substitution*

Hanoch (1975) suggests that demand models that rely on implicit additivity, such as the CDE demand model, are more appropriate for analyzing goods that are defined at a broad level. This is because these models operate under the assumption that their Allen-Uzawa partial elasticities of substitution between goods k and m are always proportionate to their partial substitution elasticities (i.e., $\sigma_{km} \propto \alpha_k$), governed by a single-good k's substitution characteristics. The assumption of implicit additivity offers insights into the relationships between *non-specific* goods, but their corresponding substitution behaviors are independent of whether the two goods are close substitutes or not (Hanoch (1975), Barten (1977)).

To illustrate this point, let us consider two sets of examples: coffee and tea vs. coffee and pastries, and smartphones and tablets vs. smartphones and laptops. In the first set, coffee and tea can be considered close substitutes due to their similar use cases and consumption patterns. On the other hand, coffee and pastries, though often consumed together, satisfy different consumer needs (beverage vs. food), making them less direct substitutes. Similarly, in the second set, smartphones and tablets are more like substitutes due to their overlapping functionalities and use cases, as opposed to smartphones and laptops, which, while "technologically" related, usually serve distinctly different purposes for the consumer.

Given the model's difficulty in precisely capturing the substitution behaviors among closely substitutable goods, we shall be cautious in how we group these goods to avoid issues in our estimation. To do so, we select aggregation categories that the model can accurately represent. Specifically, we classify into 9 aggregate categories based on the basic headings level of 109 commodities and services (see Table E).

5.2. *Expenditure Shares and Aggregate Expenditure*

As the ICP data set offers expenditure information on basic headings, we consistently aggregate them based on the 9 commodities and services across different regions. The per capita expenditure share $\Lambda_{i,k}$ for commodity k in each region i is simply calculated as per capita expenditure across demand categories $\Xi_{i,k}$ over per capita total expenditure w_i, and thus $\Lambda_{i,k} = \Xi_{i,k}/w_i$.[8] We use $w_i = \sum_k \Xi_{i,k}$ directly from the data, which represents the observed total expenditures in the indirect relationship $p_{i,k}/w_i$. Similarly, we use the observed shares $\Lambda_{i,k}$ as the estimator in the maximum likelihood estimation function. Next, we use the prices $p_{i,k}$, aggregate expenditure w_i, and model parameters to construct the fitted share $\widehat{\Lambda}_{i,k}$ by applying equations (2.2)–(2.4).

5.3. *GEKS Method for Aggregate Price Calculation*

The World Bank suggests that we utilize the GEKS method for computing prices above the basic heading level within regions (The World Bank, 2021). Diewert (2013) provides a comprehensive discussion of the method, which is based on the earlier studies of Gini (1924, 1931), Éltetö and Köves (1964), and Szulc (1964). The GEKS method is a flexible and widely used approach for estimating PPPs at different levels of aggregation. It is specifically designed to maintain the *characteristicity* of binary comparisons by using a geometric mean-based ap-

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[8]The per capita expenditure is calculated by dividing the aggregate expenditure by the population and then adjusting for the exchange rate using the United States as the reference currency. In our later computation of the aggregate price above the basic heading, we maintain consistency by choosing the United States as the numeraire.

proach to aggregate the pairwise comparison matrices.[9] For our purpose, the primary benefit of employing the GEKS method is its capability to handle a vast quantity of commodities and countries, allowing for the estimation of consistent price variations for a broad spectrum of goods and services at the aggregate level.

As the GEKS method requires the calculations of both the *Laspeyres* index and the *Paasche* index as a preliminary step, we first construct a quantity index by using the ICP price index (at the level of basic headings, g) as follows:

$$q_{i,g} = \frac{\Xi_{i,g}}{p_{i,g}}, \tag{5.1}$$

where $q_{i,g}$, $\Xi_{i,g}$, and $p_{i,g}$ refer to the quantity, expenditure, and price of commodity g in region i, respectively. By choosing basic heading PPPs for price variations within and between regions, we obtain the corresponding quantity index for calculating the Fisher (1922) index P^{Fisher} between region i relative to i', which is defined as the multilateral geometric mean of the Laspeyres price index and Paasche price index between countries i and i':

$$P_k^{Fisher}(p_i, p_{i'}, q_i, q_{i'}) = \left[P_k^{Laspeyres}(p_i, p_{i'}, q_i, q_{i'}) \cdot P_k^{Paasche}(p_i, p_{i'}, q_i, q_{i'}) \right]^{0.5}, \tag{5.2}$$

where

$$P_k^{Laspeyres}(p_i, p_{i'}, q_i, q_{i'}) \equiv \left(\sum_{g \in \mathcal{G}_k}^{n} p_{i',g} \cdot q_{i,g} \right) \Bigg/ \left(\sum_{g \in \mathcal{G}_k}^{n} p_{i,g} \cdot q_{i,g} \right), \tag{5.3}$$

and

$$P_k^{Paasche}(p_i, p_{i'}, q_i, q_{i'}) \equiv \left(\sum_{g \in \mathcal{G}_k}^{n} p_{i',g} \cdot q_{i',g} \right) \Bigg/ \left(\sum_{g \in \mathcal{G}_k}^{n} p_{i,g} \cdot q_{i',g} \right), \tag{5.4}$$

where we use k to denote an aggregated commodity; for each k, there exists a set $\mathcal{G}_k$ that contains the basic heading indices that compose the aggregate commodity k, and expenditures across aggregate goods $\Xi_k = \sum_{g \in \mathcal{G}_k} \Xi_g$. The index sets $\mathcal{G}_k$ are a partition of the basic headings, i.e. the union of these non-overlapping index sets is the set of all basic headings.

There are multiple justifications for using the Fisher index (Fisher, 1922), including the search for the 'optimal' symmetric average between the Laspeyres and Paasche indexes, the axiomatic or test approach to index number theory (in constructing price indexes), and Diewert's economic (optimization) approach to index number theory (Diewert (1988), Balk (2008), Diewert (2013)). Following Diewert (2013), we compute the aggregate PPPs as follows:

$$P_{i',k} = \prod_{i=1}^{I} \left[P_k^{Fisher}(p_i, p_{i'}, q_i, q_{i'}) \right]^{\frac{1}{I}}, \quad i' = 1, ..., I. \tag{5.5}$$

Lastly, we obtain the aggregate PPPs using equation (5.5). From these, we select a numeraire $\delta = P_{USA}$ such that the real PPPs, adjusted for the numeraire, are defined as:

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[9]Characteristicity refers to the idea that binary comparisons should be transitive, i.e., $A \succeq B \succeq C \rightarrow A \succeq C$ $\forall A, B, C \in Z$. In other methods for comparing prices across countries, the characteristicity of binary comparisons can be violated due to inconsistencies in the pairwise comparisons.

$$p_{i,k} \equiv \widetilde{P}_{i,k} = P_{i,k}/\delta_k, \quad i = 1, ..., I, \tag{5.6}$$

where $\widetilde{P}_i$ represents the amount of country i's currency needed to purchase one unit of U.S. currency while receiving an equivalent amount of utility (Diewert, 2013). To check the robustness of our prices calculation, we apply the approach outlined in Diewert (2013), which involves computing the aggregate price index by using the share-weighted *arithmetic* mean of relative prices for the Laspeyres price index for country i, and the share-weighted *harmonic* mean of relative prices for the Paasche price index for country i'. We find that the results are identical to $\widetilde{P}$ calculated from equation (5.6).

5.4. *Take Aways from the Compiled Data*

Before proceeding with the estimation, we take a closer look at the data. We review the summary statistics following the calculation of price variations using the procedure discussed in Section 5.3. Appendix F shows summary statistics of prices, quantities, per capita expenditures across commodities, expenditure shares, and per capita income. An overview of Table F.I shows that the prices for *Miscellaneous Goods and Services* are generally lower than those for food categories such as *Bread and Cereals*, and *Meat, Seafood, and Dairy*. This difference in price levels could be attributed to several factors. For example, *Miscellaneous Goods and Services* category is composed of a wide array of products and services, which may have varying price levels. Some items within this category, such as newspapers, books, stationery, and small personal care products, may have lower prices compared to food items (see Table E).

An issue with the GEKS approach is that it assigns equal importance to countries with vastly different levels of development and distinct relative prices, as it does to countries that share similar developmental stages and relative price structures (World Bank, 2013). To address the issue of weighting countries with different levels of development and relative prices, alternative methods or adjustments could be applied. For example, a weighted GEKS method can be employed, which takes into account the size of the economies, population, or GDP. Another approach could be the use of regional or income group-specific comparisons, which group countries with similar characteristics together to minimize the effect of development stage differences on the price comparisons. As such, dealing with aggregate prices in empirical literatures have always been a challenge. Our method has considered the characteristicity of binary comparisons and multilateral PPPs that maintain transitivity and base-country invariance. Finally, the prices used in the estimation are effectively transformed into *per-capita-income-weighted* or per-capita-income-normalized prices, denoted as $p_{ik}L_i/Y_i = p_{ik}/w_i$, where L_i represents the regional population, and Y_i refers to the regional income. This specification takes into account both the population size and income level in each region, thus providing a more accurate representation of the overall price structure.

Figure (G.1) shows the per capita expenditure shares of the aggregate commodity categories for different income groups. We see that the proportion of expenditure on "necessities" is lower in the 30 richest countries compared to other income groups, including the world average. Additionally, the expenditure data reveals that 30 middle-income countries allocate a higher share of their expenditure on *bread and cereals* than the 30 lowest-income countries, who allocate a greater share to *meat, seafood, and dairy*. These observations contradict the typical pattern implied by the theory of *Engel's curve*, which suggests that as income increases, the proportion of expenditure on subsistence decreases. This may suggest that due to the extreme range of income levels and level of aggregation that the composition of these goods may change with income, with *bread and cereals* representing staples in more basic forms in low income countries and more highly processed products in high income countries.

6. ESTIMATION RESULTS

Table I presents the parameter estimates of the CDE demand system, which are the parameter values that maximize the likelihood function (4.1). To carry out the estimation, we formulate it as a mathematical programming problem using the General Algebraic Modeling System (GAMS) version 40.4, a flexible modeling platform for advanced decision-making, optimization, and simulation. We employ the CONOPT nonlinear programming (NLP) solver (Drud, 1985) on a Windows 64-bit operating system to solve the problem.[10] GAMS offers a high-level, intuitive interface that facilitates the definition and manipulation of complex models, while the CONOPT solver ensures the efficient and accurate resolution of the estimation problem. This combination allows us to effectively maximize the likelihood function and obtain the desired parameter estimates for the CDE demand system.

6.1. *Bootstrapping*

To assess the robustness of the estimates, we use a paired sampling technique rather than a residual bootstrap approach. Given the constraint of expenditure shares (summing to one) on the response variable, it is more reasonable to use sample pairs of $\{\Lambda, \mathbf{p}/w\}$—with replacement—rather than residuals, as the latter may lead to negative shares. In this context, $\mathbf{p}/w$ represents the vectors of explanatory variables corresponding to unit-cost prices. Drawing Λ's directly ensures the attainment of appropriate expenditure share values. We re-estimate the model using this bootstrap re-sampled data matrix and repeat the process 10,000 times to obtain the empirical distribution of bootstrapped estimates. We present the bootstrapped empirical confidence bounds (along with point estimates) in Table I. The empirical distribution is reassuring, as most bootstrapped distributions tend to exhibit normality. Owing to the laws of large numbers and the asymptotic normality of the ML estimator, we expect these distributions to be normal. In rare instances where a bootstrapped distribution appears more uniform than normal, it may indicate that the estimator suffers from small-sample bias.

Appendix C also shows the income elasticity results by good over the range of observed income. To facilitate comparison across income levels, these elasticities are evaluated at mean prices for each country's income level. In addition to the point estimates of the elasticites, the lower and upper 90% confidence bounds are also displayed.

We discuss numerical scaling and economic robustness in Appendix C, where we assess the robustness of parameters against changes to the right-hand side of the defining equation 2.1, and examine economic robustness by calculating income and price elasticities. We find that, while parameter estimates change somewhat with the defining equation's right-hand side, the income and price elasticities do not change substantially. In calculating the demand elasticities, we develop a procedure to estimate fitted expenditure shares and utilities with mean prices. These are displayed graphically in Figure G.2, revealing that while the income elasticiites change rapidly at low income levels, the changes at higher income levels are much more gradual. In Appendix D, we develop a method to evaluate the latitude for changing parameters to determine if additional parameter normalizations are necessary, concluding that there is little latitude for parameter change given the normalizations imposed.

Our income elasticity estimates shown in Table C.2 (Appendix C.2) are in line with those from a recent 'demand-in-international-trade' study by Caron, Fally, and Markusen (2014). While their analysis provides detailed, disaggregate-level estimates using the GTAP database

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[10]The work by Drud (1985) has been widely cited and applied in numerous studies within the fields of engineering and economics. We select CONOPT4, the most advanced version of the CONOPT NLP solver available in GAMS, due to its efficiency, reliability, reproducibility, and its status as open-source software.

TABLE I

ESTIMATED CDE PARAMETERS WITH BOOTSTRAPPED EMPIRICAL CONFIDENCE BOUNDS[*]

Consumption bundle	α	β	e
Bread and cereals	0.942	0.032	0.270
	(0.027)	(0.039)	(0.096)
	[0.894; 0.960]	[0.010; 0.067]	[0.088; 0.419]
Meat, seafood and dairy	0.544	0.001	1.063
	(0.310)	(0.120)	(0.327)
	[0.000; 0.966]	[0.000; 0.303]	[0.642; 1.636]
Calorie-dense foods	0.530	0.001	1.047
	(0.218)	(0.047)	(0.197)
	[0.201; 0.896]	[0.000; 0.020]	[0.661; 1.301]
Fruit and vegetables	0.859	0.008	0.584
	(0.193)	(0.068)	(0.181)
	[0.371; 0.961]	[0.000; 0.146]	[0.447; 1.028]
Textiles and apparels	0.325	0.000	1.423
	(0.247)	(0.128)	(0.214)
	[0.000; 0.854]	[0.000; 0.008]	[0.935; 1.660]
Household utilities	0.641	0.005	1.385
	(0.119)	(0.096)	(0.122)
	[0.460; 0.851]	[0.002; 0.027]	[1.163; 1.558]
Manufactured goods	0.942	0.044	0.973
	(0.074)	(0.223)	(0.087)
	[0.762; 0.986]	[0.003; 0.704]	[0.865; 1.142]
Transport and communications	0.976	0.763	0.997
	(0.105)	(0.317)	(0.131)
	[0.666; 0.978]	[0.003; 0.887]	[0.894; 1.332]
Miscellaneous services	0.928	0.147	1.258
	(0.075)	(0.249)	(0.104)
	[0.754; 0.978]	[0.018; 0.846]	[1.110; 1.449]

[*]Point estimates obtained from paired bootstrap with 10,000 replicates; 90% confidence intervals displayed in square brackets; standard errors in parentheses.

derived from the supply-side data, our study uses broader categories based on consumer prices and expenditures. For instance, they report high income elasticities for specific items like *wearing apparel* at 1.057 and *wool, silk-worm cocoons* at 1.426. In comparison, our broader *textile and apparel* category shows a similarly high elasticity of 1.21. This implies that despite the differences in data sources and granularity, we both observe a trend where spending on textiles and apparel increases significantly with income.

Furthermore, both studies find that transportation, communication, and financial services have relatively lower income elasticities compared to *textile and apparel*. For example, Caron, Fally, and Markusen (2014) note an elasticity of 1.152 for communication, with business and financial services showing higher elasticity than communication but still lower than those for *textile and apparel*. Our findings have similar trends for these categories, with *transportation and communication* at 0.97 and *miscellaneous services* at 0.99. Our *bread and cereals* category appears higher at 0.93, which again reflects consumer expenditures on processed foods like *bread* with consumer-facing prices. This is somewhat analogous to their estimates for *Wheat* (0.883) and *Crops (not elsewhere classified)* (0.893).

Again, as discussed in Appendix C.2, our approach, which adjusts for average prices and uses fitted shares and utility, allows for more adaptable and consistent predictions of how consumption patterns change with income, offering a complement to the median expenditure method used by Caron, Fally, and Markusen (2014).

7. CONCLUSION

Preference structures in applied general equilibrium models are often limited to constant-elasticity-of-substitution or CES forms due to the desire for global regularity. Hanoch (1975) uses implicit, additive relationships, that can be viewed as a generalization of the CES, to obtain more flexible demand relationships. These preference relationships separate substitution effects from income effects in ways that go beyond relaxation of homotheticity. However, the estimation of these models as demand systems has proven to be difficult and most published work in this area has focused on approaches that involve approximations.

We contribute to the literature by directly estimating this demand system for the first time. We exploit tools for constrained numerical optimization, which allows us to perform direct estimation within a maximum likelihood framework. In doing this, we find that the global regularity conditions stated in Hanoch (1975) can be slightly relaxed, and that the relaxed parametric conditions facilitate estimation. We introduce a normalization scheme that is beneficial for the scaling of the parameter values and which appears to have little impact on the economic performance of the estimated system. Our central finding is that the direct estimation of this type of demand system is tractable and practical. While critics may object to the fact that we estimate the unobservable utility levels, we argue that we do so no more than those who estimate standard CES functions. That is, econometricians estimate all of the parameters necessary to *evaluate* utility, and so may as well have estimated utility. Because the system we estimate is implicit, we have no choice but to explicitly estimate utility.

On the investigation of the robustness of our parameter estimates, we use a series of numerical tests to verify that the parameter values cannot be changed by an economically significant amount without reducing the likelihood function, suggesting that additional normalizations are not needed for parameter identification in this model. Thus our estimation procedure appears to be computationally feasible and the parameter values identified in the context of a direct maximum likelihood estimation of the Constant Difference of Elasticity preference relationship.

Drawing inspiration from the seminal work of Berry (1994) and Berry, Levinsohn, and Pakes (1995), our approach capitalizes on "share information" to estimate unobservable utilities. By leveraging a numerical constrained optimization procedure, we are able to explicitly derive utility levels from an implicit additive indirect demand system, while still maintaining transparency and tractability in a structural procedure. Our approach allows identification of all model parameters, leading to estimates of a full set of demand elasticities. A future direction for our research may be to incorporate instrumental variables into our procedures as in Berry (1994), and the application of this approach to estimating other general demand models that require the calculation of unobservables.

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APPENDIX A: THE PROOF OF MONOTONICITY OF THE CRES

PROOF: For the convenience of the reader we restate the CRES function here:

$$\mathscr{H}(\mathbf{x}, u) = \sum_k \beta_k u^{-e_k(1-\alpha_k)} (x_k)^{1-\alpha_k} \equiv 1, \tag{A.1}$$

where (i) $\beta_k, e_k > 0 \ \forall \ k \in N$, and (ii) either $\alpha_k > 1$ or $0 \leq \alpha_k < 1 \ \forall \ k \in N$. (Note that this argument can be patched to handle $0 \leq \alpha_k \leq 1$ by dealing with cases. However, the fact that, when we interpret $x_k^{1-\alpha_k} = \ln(x_k)$ when $\alpha_k = 1$, the derivative of $x_k^{1-\alpha_k}$ with respect to x_k remains positive, means that the argument below goes through.)

Taking the total differential of (A.1) to get:

$$-\sum_k \beta_k e_k (1-\alpha_k) u^{-e_k(1-\alpha_k)-1} x_k^{1-\alpha_k} du + \sum_k \beta_k u^{-e_k(1-\alpha_k)(1-\alpha_K)} x_k^{-\alpha_k} dx_k \equiv 0. \tag{A.2}$$

Now let all $dx_k = 0$ except for $k = m$ to derive:

$$-\sum_k \beta_k e_k (1-\alpha_k) u^{-e_k(1-\alpha_k)-1} x_k^{1-\alpha_k} du + \beta_m u^{-e_m(1-\alpha_m)} (1-\alpha_m) x_m^{-\alpha_m} dx_m \equiv 0. \tag{A.3}$$

Then we solve for the change in u for a change in x_m:

$$\frac{du}{dx_m} = \frac{\beta_m u^{-e_m(1-\alpha_m)}(1-\alpha_m)x_m^{-\alpha_m}}{\sum_k \beta_k e_k(1-\alpha_k)u^{-e_k(1-\alpha_k)-1}x_k^{1-\alpha_k}}. \tag{A.4}$$

Notice that the numerator has the sign of $(1-\alpha_m)$, and every term in the sum in the denominator has the sign of $(1-\alpha_k)$. These signs are the same by the parametric restrictions, and hence:

$$\frac{du}{dx_m} > 0. \tag{A.5}$$

Thus, u is strictly monotonic in each x_k. *Q.E.D.*

APPENDIX B: SOLUTIONS TO THE EXPENDITURE MINIMIZATION PROBLEM UNDER THE RELAXED REGULARITY CONDITIONS

The relaxation of the regularity conditions for the CRES demand system has impacts on the nature of the solutions to the expenditure minimization problem. Again we focus on the regime where $0 \leq \alpha_k < 1$, and focus on the difference between whether zero is included as the minimum value. Under the original conditions where $\alpha_k > 0$, solutions to the utility maximization problem satisfy $x_k > 0$. If $\alpha_k = 0$ is permitted for some subset of goods, the possibility that $x_k = 0$ cannot be excluded. Here we characterize optimal solutions for that case.

We define K_0 to be the set of indices for goods with $\alpha_k = 0$, and K_+ to be the subset of indices with $\alpha_k > 0$. It is convenient to consider two cases with the first being when the solution for the expenditure minimization problem has $x_k = 0$ for all $k \in K_0$, and the second being when the solution has $x_k > 0$ for one or more $k \in K_0$.

Case 1 ($x_k = 0$ for all $k \in K_0$):

Consider the first-order conditions for x_k when $k \in K_+$. These are:

$$c_k - \lambda\beta_k u^{-e_k(1-\alpha_k)}(1-\alpha_k)x_k^{-\alpha_k} = 0. \tag{B.1}$$

The case where $x_k = 0$ cannot occur because the second term on the left-hand side of the above decreases without bound as x_k approaches zero. Solving for x_k:

$$x_k = [c_k/(\lambda\beta_k u^{-e_k(1-\alpha_k)}(1-\alpha_k))]^{-1/\alpha_k} \tag{B.2}$$

for $k \in K_+$ and set $x_k^* = 0$ for $k \in K_0$. Substituting these into the defining constraint:

$$\sum_{k \in K_{0+}} \beta_k u^{-e_k(1-\alpha_k)}[c_k/(\lambda\beta_k u^{-e_k(1-\alpha_k)}(1-\alpha_k))]^{-1/\alpha_k} = 1. \tag{B.3}$$

This equation can be uniquely solved for λ, and the result can be substituted back into (B.2) to determine the solution for x_k. To verify that this is the correct case, it only remains to verify that the first-order conditions for x_k for $k \in K_0$ are satisfied:

$$c_k - \lambda\beta_k u^{-e_k} \geq 0, \tag{B.4}$$

for all $k \in K_0$. If these conditions are satisfied, then set $x_k = 0$ for all $k \in K_0$, and the solution is defined. If one or more of these conditions are violated, it must be that $x_k > 0$ for some $k \in K_0$, and Case 2 defines the solution.

Case 2 ($x_k > 0$ for some $k \in K_0$):

Consider the first-order conditions for x_k when $k \in K_0$. These are:

$$c_k - \lambda \beta_k u^{-e_k} \geq 0, \tag{B.5}$$

and

$$x_k[c_k - \lambda \beta_k u^{-e_k}] = 0. \tag{B.6}$$

Since at least one $x_k > 0$ for $k \in K_0$, the value of λ calculated in Case 1 is not corrrect. The correct value can be calculated as:

$$\lambda = \min_{k \in K_0} \left\{ c_k \left[\beta_k u^{-e_k(1-\alpha_k)} \right]^{-1} \right\}. \tag{B.7}$$

The index set K_{0+} is defined as the $k \in K_0$ that achieve the minimum in (B.7). Using this λ, the optimal values for x_k where $k \in K_+$:

$$x_k = [c_k/(\lambda \beta_k u^{-e_k(1-\alpha_k)}(1-\alpha_k))]^{-1/\alpha_k}, \tag{B.8}$$

and set $x_k = 0$ for all k in K_0 but not in K_{0+}. Any values of the $x_k \geq 0$ for $k \in K_{0+}$ that satisfy the following equation will be optimal:

$$\sum_{k \in K_{0+}} \beta_k u^{-e_k} x_k = \sum_{k \in K_+} \beta_k u^{-e_k(1-\alpha_k)} x_k^{1-\alpha_k} + 1. \tag{B.9}$$

This completes the characterization of the solution to the expenditure problem. Relaxing the lower bound on α_k is important for parameter estimation, where we have found that this bound may be active at times.

APPENDIX C: NUMERICAL SCALING AND ECONOMIC ROBUSTNESS

Previous empirical framework of implicitly direct demand systems, such as Cranfield et al. (2002) and Preckel, Cranfield, and Hertel (2010), note that the right-hand side of the implicit additivity defining equation can equal to any constant $\mathscr{M} \in \mathbb{R}$ and that it can be estimated. They proceed to estimate its numerical value. Given the excess degrees of freedom demonstrated in equation (3.7), it can be readily seen that equation $\sum_k \beta_k = 1$ is a *de facto* normalization where $\mathscr{M} \equiv 1$. In the interest of good scaling of the individual terms in the sum in the defining equation (4.4), we run a series of likelihood value testing by choosing to set:[11]

$$G(\frac{p_k}{w}, u_i) \equiv \sum_k \beta_k \equiv \mathscr{M} \in \mathbb{R}_{++} \quad \forall \ \beta_k > 0. \tag{C.1}$$

C.1. *Log-Likelihood Values*

The results presented in Table I correspond to $\mathscr{M} = 1$, where the log-likelihood function achieves a maximized objective value of 2814.42. We observe that as we increase the joint right-hand sides of the sum of β's and the defining constraint, the objective value (i.e., the log-likelihood) unambiguously increases. Specifically, as we vary $\mathscr{M}$ from 1 to 10 to 100 to 1,000,

---

[11]The parametric restrictions in this model govern that $\mathscr{M}$ must be strictly positive, since β_k's are strictly positive.

and 10,000, the objective value also increases, albeit with a progressively smaller growth rate. For example, when we increase $\mathcal{M}$ by a factor of 10 from 1 to 10, the likelihood value increases by approximately 0.09%. When we increase $\mathcal{M}$ from 10 to 100, the likelihood increase is about 0.05%, and for the increase from 100 to 1,000, the likelihood increase is less than 0.02%.

In parallel, the scaling of the model parameters suffers with the magnitude of $\mathcal{M} \in \mathbb{R}_{++}$. As $\mathcal{M}$ approaches infinity with each time increasing by a factor of 10 to ultimately achieve 10,000, algorithms for solving constrained NLP unsurprisingly require an increasingly large number of iterative steps in the parameter space (provided by the nonlinear implicit relationships) to seek an optimal solution, while the scaling of the problem becomes incrementally worse and β's eventually become extreme across goods.[12] However, if one examines the *economic properties* of the demand system, the elasticities of substitution and income elasticities do not change appreciably.

C.2. *Testing for Income Elasticity of Demand*

We use the estimated parameters to calculate the income elasticities of the CDE demand system, which is given by:

$$
\eta_k = \frac{e_k(1 - \alpha_k) + \sum_m \widehat{\Lambda}_m e_m \alpha_m}{\sum_m \widehat{\Lambda}_m e_m} + \alpha_k - \sum_m \widehat{\Lambda}_m \alpha_m, \tag{C.2}
$$

where, for any country in the dataset, η_k represents the income elasticity of commodity k, and $\widehat{\Lambda}_k$ denotes the estimated expenditure share for that commodity (note that subscripts for country have been omitted for simplicity).

In demand estimation literature, researchers may choose between fitted and initial expenditure shares based on their objectives. While initial shares are suitable for assessing model fit or comparing demand models, our focus in this section is on accurately representing consumer preferences and behavior associated with different incomes. By using fitted shares instead of baseline shares, we tolerate potential deviations from baseline shares and enable consistent model predictions of consumption trajectories that evolve with changes in income levels.

Our calculating procedure involves fitting the utilities based on average prices. We formulate a constrained maximization program with the objective of maximizing *aggregate welfare* across regions, while holding the estimated model parameters, $\widehat{\alpha}$, $\widehat{\beta}$, and $\widehat{e}$, fixed:

$$
\begin{aligned}
\text{maximize} \quad & \sum_i u_i \\
\text{subject to:} \quad & \sum_k \widehat{\beta}_k u_i^{\widehat{e}_k(1 - \widehat{\alpha}_k)} \left(\frac{\bar{p}_k}{w_i} \right)^{1 - \widehat{\alpha}_k} - 1 \equiv 0,
\end{aligned} \tag{C.3}
$$

where $\bar{p}_k = \sum_i p_{ik}/N$ are average prices $\forall\, k$.

---

[12]To reduce the computational burden in the initial phase of minimizing infeasibility, we first estimate a parameterized implicit non-homothetic CES model by setting $\alpha_k = \alpha \,\forall\, k$ (i.e., an implicit NHCES Model; see also Comin, Lashkari, and Mestieri (2021). Yang (2021) shows this parameterization and that the *quasi* Marshallian correspondence of the parameterized indirect case is identical to the direct case in Comin, Lashkari, and Mestieri (2021). This procedure is effective for finding starting values that are feasible to characterize the implicit indirect relationships in the estimation of the CDE.

Using average prices to calculate income elasticities across countries simplifies the process compared to using individual prices, avoiding potential errors and inconsistencies from possible multilateral comparison biases and changing *economic structures*.[13] With a single reference point for price levels, average prices allow clearer cross-country comparisons and focus on the impact of income differences on demand and consumption behavior.

We calculate fitted shares using equations (2.3) and (2.4) and then employ the estimated parameters and these fitted shares to compute income elasticities with equation (C.2). Upon calculating average income elasticities across commodities, we observe that as $\mathscr{M}$ increases from 1 to 10,000, the absolute percentage changes in income elasticities for most goods, when compared to the base where $\mathscr{M} \equiv 1$, remain less than 1% (see Table C.I). It should be noted that the absolute differences between income elasticities are mostly negligible, with some being practically zero up to the hundredth decimal place, even when the joint right-hand-side values of the sum of β's are shifted to 10,000.

TABLE C.I

ROBUSTNESS OF INCOME ELASTICITIES

Commodities	$\mathscr{M} = 1$	$\mathscr{M} = 10,000$	Change (%)	Abs. Change
Breadandcereals	0.93	0.97	3.84	0.04
MeaSeaDar	0.98	0.93	-4.53	-0.04
Caldense	0.97	0.97	0.30	0.00
FruitVegie	0.91	0.95	3.64	0.03
TextAppar	1.21	1.29	7.41	0.09
HousUtils	1.08	1.07	-1.32	-0.01
Mnfcs	0.97	0.97	0.20	0.00
TransComm	0.97	0.97	-0.13	0.00
MiscServ	0.99	0.97	-1.46	-0.01

Note: Values are rounded to two decimal places.

C.3. *Testing for Own-Price Elasticity of Demand*

The Allen-Uzawa elasticities of substitution σ_{km} is given by:

$$\sigma_{km} = \alpha_k + \alpha_m - \sum_n \Lambda_n \alpha_n - \frac{\delta_{km} \alpha_k}{\Lambda_k}, \tag{C.4}$$

where δ_{km} is the Kronecker delta (equals 0 if $k \neq m$, 1 if if $k = m$).

The uncompensated price elasticities of substitution ϵ_{km}^{unc} is given by

$$\epsilon_{km}^{unc} = (\sigma_{km} - \eta_k) \widehat{\Lambda}_k. \tag{C.5}$$

Taking into account both the substitution effect and the income effect, we examine the changes in uncompensated (Marshallian) own-price elasticities and find that altering $\mathscr{M} \in \mathbb{R}_{++}$ from unity has minimal impact on economic behavior. The initial own-price elasticity values are

---

[13]*Economic structure* refers to variations in the composition across economies over time. It may include factors such as changes in product characteristics or preferences across commodities, the distribution of income, or the structure of trade relationships. As economies evolve and change, these factors can also shift, potentially affecting the consistency of cross-country comparisons when using the ICP price index.

small, with minor percentage changes even as $\mathcal{M}$ approaches 100,000. For instance, changes in uncompensated own-price elasticities for most commodities are less than 10% as $\mathcal{M}$ increases from 1 to 10,000. The percentage changes are relatively larger for meat, seafood, and dairy products, but their initial base values are not substantial enough to cause significant impacts on economic performance when considering the absolute changes from the base value of the joint right-hand sides where $\mathcal{M} \equiv 1$.

For example, *Meat, Seafood, and Dairy* has an average uncompensated own-price elasticity of about -0.61. Even as $\mathcal{M}$ increases to 10,000 (a factor of 10,000), the absolute change in compensated (uncompensated) own-price elasticities of *Meat, Seafood, and Dairy* is only about -0.19. This observation suggests that variations in $\mathcal{M}$ have limited impacts on the overall economic behavior as reflected by the uncompensated own-price elasticities (see Table C.II).

TABLE C.II

ROBUSTNESS OF UNCOMPENSATED OWN-PRICE ELASTICITIES

Commodities	$\mathcal{M} = 1$	$\mathcal{M} = 10,000$	Change (%)	Absolute Change
Breadandcereals	-0.93	-0.98	5.59	-0.05
MeaSeaDar	-0.61	-0.80	31.84	-0.19
Caldense	-0.59	-0.63	7.53	-0.04
FruitVegie	-0.86	-0.93	8.65	-0.07
TextAppar	-0.39	-0.34	-11.43	0.04
HousUtils	-0.75	-0.78	5.15	-0.04
Mnfcs	-0.94	-0.99	5.65	-0.05
TransComm	-0.95	-0.97	2.68	-0.03
MiscServ	-0.91	-0.95	4.84	-0.04

Note: Values are rounded to two decimal places.

APPENDIX D: LATITUDE FOR CHANGING PARAMETERS

We develop a *post-hoc* structural method to determine the latitude for changing parameters. In considering the potential for identification of the parameters of our preference relationship, we were able to demonstrate that at least two normalizations were required to remove excess degrees of freedom from the system. A reasonable follow-on question might be, are there additional normalizations that should be added? To this end, we would like to know whether the same log likelihood level can be achieved with different configurations of parameter values. While this is an exceedingly difficult question to answer absent an assurance that the problem is a convex program, we ask this question starting from the estimated parameter solution.

Operationally, we proceed as follows: First, we estimate parameters using equations (4.1)-(4.4), auxiliary identities (2.2)-(2.4), parametric restrictions in Section 4.1, and normalizations $\sum_k \beta_k = 1$ and $\sum_k e_k = N$. We then construct a new problem encompassing the original estimation problem's relationships and ensuring the likelihood function value is at least as high as the maximum obtained during parameter estimation. With these constraints, we formulate new problems to maximize and minimize each parameter individually. For example, we define a new objective variable $z \equiv \alpha_{Mnfcs}$ and solve two problems: one maximizing z and one minimizing z. All other parameters can change to accommodate z adjustments, subject to constraints. If the maximum absolute change in z compared to the original parameter estimate is virtually zero, we conclude that it is locally impossible to change the parameter without reducing the likelihood function. We repeat this procedure for all model parameters, α_k, β_k, e_k, and each country's estimated utility level.

Most parameters have a movement scope of about 0.001 (0.01 at most), corresponding to a tenth to one percent range. These findings indicate that substantial changes to parameter values cannot occur without decreasing the likelihood value or violating the problem constraints, suggesting no additional parameter normalizations are required.

APPENDIX E: COMMODITY AGGREGATION REFERENCE TABLE

TABLE E.I: Commodity Aggregation Based on the Unpublished ICP Data

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Bread and Cereals
Rice; Other cereals flour and other cereal products; Bread; Pasta products and couscous.

Meat Seafood and Dairy
Beef and veal; Pork; Lamb mutton and goat; Poultry;
Other meats and meat preparations; Fresh chilled or frozen fish and seafood; Preserved or processed fish and seafood; Fresh milk; Preserved milk and other milk products; Cheese and curd; Eggs and egg-based products.

Calorie Densed Products and Alcohol Tobacco Narcotics
Butter and margarine; Other edible oils and fats; Sugar; Jams marmalades and honey;
Confectionery chocolate and ice cream; Food products n.e.c.; Spirits; Wine; Beer; Tobacco; Narcotics

---

Fruit, Vegetables, and non-alcoholic beverages
Fresh or chilled fruit; Frozen preserved or processed fruit and fruit-based products; Fresh or chilled vegetables other than potatoes and other tuber vegetables; Fresh or chilled potatoes and other tuber vegetables; Frozen preserved or processed vegetables and vegetable-based products; Coffee tea and cocoa; Mineral waters soft drinks fruit and vegetable juices.

Textile and Apparel
Clothing materials other articles of clothing and clothing accessories; Garments; Cleaning repair and hire of clothing; Shoes and other footwear; Repair and hire of footwear.

Household Utilities
Actual rentals for housing; Imputed rentals for housing; Maintenance and repair of the dwelling; Water supply; Miscellaneous services relating to the dwelling; Electricity Gas Other fuels.

Manufactured Goods and Services
Furniture and furnishings; Carpets and other floor coverings; Repair of furniture furnishings and floor coverings; Household textiles; Major household appliances whether electric or not; Small electric household appliances; Repair of household appliances; Glassware tableware and household utensils; Major tools and equipment; Small tools and miscellaneous accessories; Non-durable household goods; Domestic services Household services.

Transport and Communication
Motor cars Motor cycles; Bicycles Animal drawn vehicles Fuels and lubricants for personal transport equipment; Maintenance and repair of personal transport equipment; Other services in respect of personal transport equipment; Passenger transport by railway; Passenger transport by road; Passenger transport by air; Passenger transport by sea and inland waterway; Combined passenger transport; Other purchased transport services; Postal services; Telephone and telefax equipment; Telephone and telefax services

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Miscellaneous Goods and Services

Pharmaceutical products; Other medical products; Therapeutic appliances and equipment; Medical services; Dental services; Paramedical services; Hospital services; Audio-visual photographic and information processing equipment; Recording media Repair of audio-visual photographic and information processing equipment; Major durables for outdoor and indoor recreation; Maintenance and repair of other major durables for recreation and culture; Other recreational items and equipment; Garden and pets; Veterinary and other services for pets; Recreational and sporting services; Cultural services Games of chance; Newspapers books and stationery Package holidays; Education - HHC; Catering services; Accommodation services; Hairdressing salons and personal grooming establishments; Appliances articles and products for personal care; Prostitution; Jewellery clocks and watches; Other personal effects Social protection - HHC; Insurance; Financial Intermediation; Services Indirectly Measured; Other financial services n.e.c.; Other services n.e.c.

APPENDIX F: SUMMARY STATISTICS

TABLE F.I

SUMMARY STATISTICS FOR PRICES

	BreadCereal	MeaSeaDar	Caldense	FruitVegie	TextAppar	HousUtils	Mnfcs	TransComm	MiscServ
Min	0.37	0.47	0.42	0.25	0.21	0.06	0.27	0.24	0.16
Max	2.48	1.79	1.88	2.11	1.55	1.61	1.78	1.55	1.47
Mean	0.92	0.83	0.90	0.82	0.68	0.46	0.71	0.77	0.53
Std	0.35	0.25	0.30	0.34	0.30	0.36	0.29	0.27	0.28

*Decimals are rounded to two places. Prices are normalized by US exchange rate.

TABLE F.II

SUMMARY STATISTICS FOR QUANTITIES

	BreadCereal	MeaSeaDar	Caldense	FruitVegie	TextAppar	HousUtils	Mnfcs	TransComm	MiscServ
Min	19.85	15.40	28.74	23.78	1.41	77.64	5.32	29.77	86.33
Max	735.54	2206.45	3285.13	1416.97	2219.17	12712.04	1752.07	5854.74	22497.03
Mean	240.99	557.77	509.53	448.84	425.82	2921.92	491.47	1405.46	3676.88
Std	120.11	364.67	466.00	274.94	382.04	2518.43	438.05	1256.21	3975.68

*Decimals are rounded to two places.

TABLE F.III

SUMMARY STATISTICS FOR PER CAPITA EXPENDITURES

	BreadCereal	MeaSeaDar	Caldense	FruitVegie	TextAppar	HousUtils	Mnfcs	TransComm	MiscServ
Min	26.20	9.41	22.61	13.82	0.50	17.95	2.39	20.19	22.26
Max	731.19	2393.95	3342.65	2130.23	1874.67	17217.01	2043.67	6260.26	21346.98
Mean	222.16	495.98	499.90	385.64	351.80	1854.32	432.92	1300.96	2702.57
Std	141.03	406.14	555.54	313.84	390.97	2614.51	494.00	1463.56	3968.78
Lower CI 95%	201.45	436.35	418.33	339.56	294.39	1470.43	360.38	1086.06	2119.82
Upper CI 95%	242.87	555.62	581.48	431.72	409.20	2238.22	505.45	1515.86	3285.32

*Decimals are rounded to two places.

TABLE F.IV

SUMMARY STATISTICS FOR EXPENDITURE SHARES

|	Breadandc	MeaSeaDar	Caldense	FruitVegie	TextAppar	HousUtils	Mnfcs	TransComm	MiscServ
Min	0.01	0.02	0.02	0.02	0.00	0.04	0.01	0.04	0.07
Max	0.30	0.42	0.26	0.38	0.22	0.45	0.13	0.33	0.54
Mean	0.06	0.10	0.08	0.08	0.05	0.18	0.05	0.15	0.25
Std	0.06	0.05	0.04	0.05	0.02	0.08	0.02	0.05	0.10
Lower CI 95%	0.06	0.09	0.08	0.07	0.04	0.17	0.05	0.14	0.23
Upper CI 95%	0.07	0.10	0.09	0.09	0.05	0.19	0.05	0.16	0.26

*Decimals are rounded to 2 places.

TABLE F.V

SUMMARY STATISTICS FOR INCOME

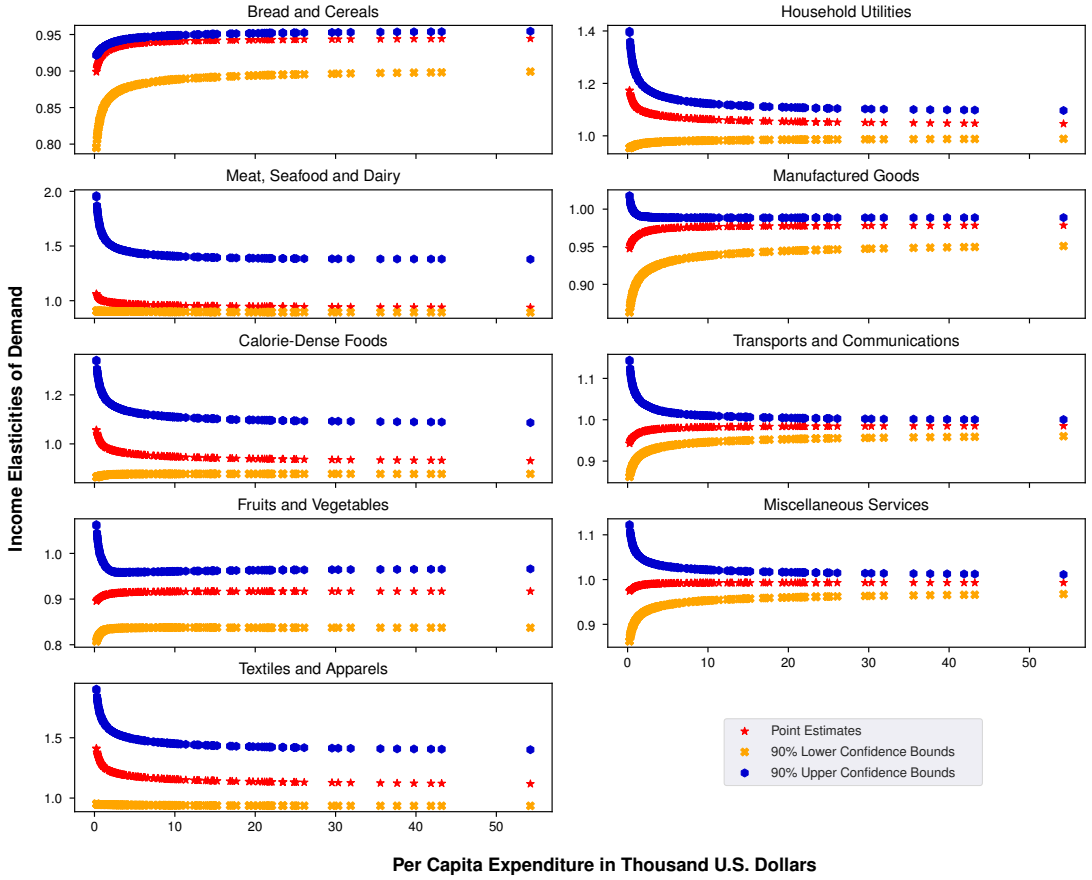|	Income
Min	236.94
Max	54236.48
Mean	8246.25
Std	9822.29
Lower CI 95%	6804.01
Upper CI 95%	9688.49

[*]Decimals are rounded to two places.

APPENDIX G: FIGURES

FIGURE G.1.—Expenditure Shares with Different Income Groups.
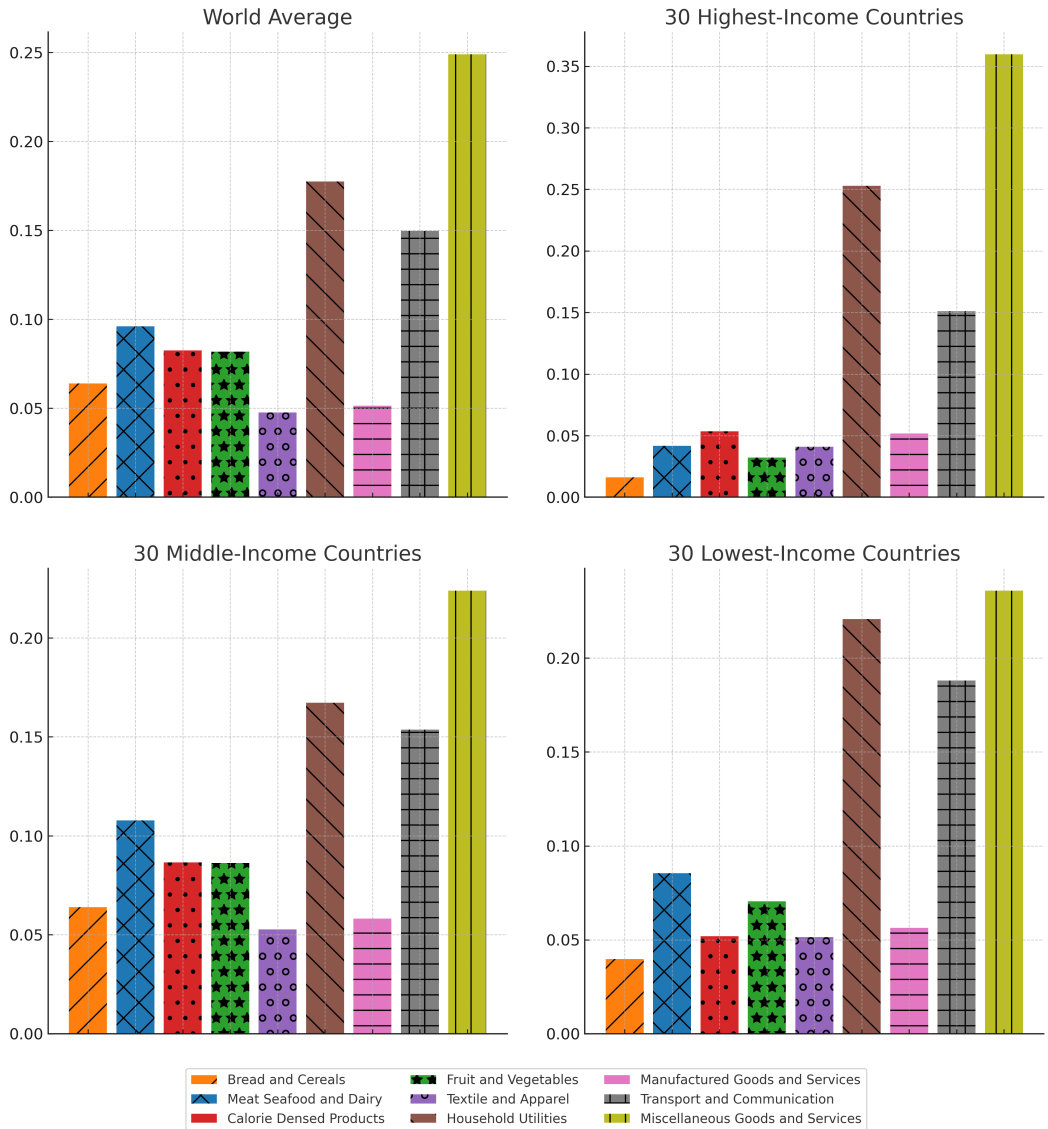
Note: The charts display expenditure shares for selected income groups, calculated from the unpublished World Bank ICP database for the reference year 2017. It is clear to see that the proportion of expenditure on "necessities" in the 30 richest countries is lower in comparison to other income groups (including the global average). Interestingly, the data shows that the 30 middle-income countries spend more on *bread and cereals* than the 30 lowest-income countries, who allocate a greater share to *meat, seafood, and dairy*. This contradicts the typical pattern implied by Engel's Law.

FIGURE G.2.—Income Elasticities versus Income.

Note: Empirical income elasticities obtained from paired bootstrap with 10,000 replicates.