Optimization with Hat Derivatives: Mathematical Concept and Derivation of an Applied Computable Partial Equilibrium Model*

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1 Background

This note links the mathematical concept of **hat derivatives** to constrained optimization results derived and used by economists which is increasingly often written in the General Equilibrium Modelling (GEMPACK) Software using the *TABLO language* of writing economic models.[1] Without these connections, it is fair to say that there are at least two substantial gaps in understanding many simple concept-based but probable lengthy economic models that are almost exclusively implemented using computers today. The first one is that, despite its importance, mathematicians almost never (very rarely, if so) formally introduce hat derivatives in a text book or publicly accessible academic sources (Maurer, 2002); the second one is that, despite its frequent use in contemporary economic computations, economists and economic modelers unwittingly lack of explaining step by step how computer languages using the concept of hat derivatives are derived from the neoclassical economic theory. Not only the gaps have long created barriers to economists' understanding of important model structures and communications among them within their own community, they sometimes also emerge from a lack of an appropriate understanding outside the

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[1]For overview of the TABLO language, see `https://www.http://www.copsmodels.com/gpmanual.htm`.

loop and thus some good quality of relevant economic research can be valued more lightly than they are supposed to be. Therefore, the purpose of this note (partially due to increasing demand for relevant inquiries) is to attempt to fill these gaps.

To do so, we use a *Simplified International Model of agricultural Prices, Land use and the Environment* (SIMPLE) Model (Baldos and Hertel, 2012)–which is an applied computable partial equilibrium model of global agricultural supply and demand–as an example; we show how equations used in this model corresponding to land use changes written in TABLO language were converted and algebraically derived from the producer theory in microeconomics. We then bring a slightly more general case and use a derivative of this model–SIMPLE-MAIDADS (Yang, Gouel and Hertel, 2018) to show derivations with interactions between long-run competition for land and multiple agricultural outputs.[2] Nevertheless, we hope that this note will have much wider adaptability and usefulness, not merely in partial equilibrium models, but as well as in general equilibrium models. For example, the sequence of steps to arrive at what has been coded in GEMPACK and their mathematical notations are highly similar to those used in the GTAP (Global Trade Analysis Project) Model, which also uses the technique of hat calculus to derive the percentage derivation from the base scenario in economic simulations (e.g., counterfactual analyses carried out on computers). At the finish, we hope that this note will be useful to users who use facilitated interface of simulated economic shocks and wish to see the theoretical foundation of its analytical framework.

2 Logarithmic Derivative

Let us denote x' by infinitesimal **absolute change** in an economic variable $x > 0$ (e.g., strictly positive quantity of consumption of fresh apples) where it is also a derivative of x, then $\log' x = x'/x = \hat{x}$ is referred to by mathematicians as infinitesimal **relative change** in x, representing the absolute change in apple consumptions relative to (scaled by) the initial quantities of apple consumption, where $\hat{x}$ is the **percentage rate of change** in x or the **instantaneous rate of relative change** to x. Mathematically, we may prove this equivalence using the limit of the average relative rate (Maurer, 2002); denote f by a function of x and that Δf is the absolute change in f, then $\hat{x} = \lim_{\Delta f \to 0} \frac{\Delta x / \Delta f}{x} = \frac{x'}{x} = \log' x$, where $\Delta x = x(f + \Delta f) - x(f)$ is absolute change in x. Therefore, the defined redux from the concept used in calculus (or in functions of complex variable) shows that the term of percentage change (from base) frequently used by economists is no different from what is widely understood by mathematicians as logarithmic derivative.

Extensions of the Percent Rate of Change Here we demonstrate some important extensions of formulae of hat derivatives used in economic models.[3]

The first one is the **(i) rule of multiplication**. If we define that

[2]MAIDADS stands for a A Modified, Implicit Directly Additive Demand System (Preckel et al., 2010).

[3]The author thanks to and benefits from Bill Gibson who summarized basic formulae of hat calculus.

$$z = xy,$$

then

$$\begin{aligned} \log z &= \log(xy) \\ &= \log x + \log y, \end{aligned}$$

and therefore

$$\begin{aligned} \log' z &= \log'(xy) \\ &= \log' x + \log' y, \end{aligned}$$

which is equivalent as stating

$$\frac{z'}{z} = \frac{x'}{x} + \frac{y'}{y},$$

or [4]

$$\hat{z} = \hat{x} + \hat{y}.$$

Certainly, if we define c as a constant and replace it with the variable x (that is, if $z = cy$), then by the **(ii) rule of constant multiplication**, we have

$$\hat{z} = \hat{y}.$$

Following the same logic, we may write **(iii) the rule of division**. If we have that

$$z = \frac{x}{y},$$

then

$$\hat{z} = \hat{x} - \hat{y}.$$

Finally, the **(iv) rule of exponents** states that, if $z = x^b$, then

$$\hat{z} = b\hat{x}.$$

[4] Same results can be derived using the product rule, e.g., $\hat{z} = \widehat{xy} = \hat{x} + \hat{y}$.

3 Economics: Land Use in the Standard SIMPLE

We now apply hat derivatives and its extensions and rules in practical economic problems. To benefit users who read TABLO codes and/or use AnalyseGE to evaluate and decompose changes in endogenous variables based on changes in exogenous shocks, we adopt variable names that are close to what have been written in the TABLO file.

Farmer's (Constant Elasticity of Substitution) CES production technology in the standard SIMPLE is modeled as

$$Q_{crop} = AO_{crop}[(AF_{land}Q_{land})^{\rho} + (AF_{nonland}Q_{nonland})^{\rho}]^{\frac{1}{\rho}}. \tag{1}$$

where AO_{crop} is the Hicks-neutral efficiency index (or Total Factor Productivity) in crop production; AF_{land} is the land-biased efficiency index in crop production; $AF_{nonland}$ is the non-land input biased efficiency index in crop production; Q_{crop} is the output quantity of crops; Q_{land} and $Q_{nonland}$ are quantities of land and non-land inputs, respectively; ρ is a CES substitution parameter with $\frac{1}{1-\rho}$ equaling σ.

Farmers minimize expenditure and are subject to technology constraints

$$\mathcal{L} = P_{land}Q_{land} + P_{nonland}Q_{nonland} + \lambda[\overline{Q}_{crop} - Q_{crop}] \tag{2}$$

First-Order Condition (F.O.C.) gives that

$$\begin{aligned}
&P_{land} = \lambda\frac{AO_{crop}}{\rho}[(AF_{land}Q_{land})^{\rho} + (AF_{nonland}Q_{nonland})^{\rho}]^{\frac{1-\rho}{\rho}}AF^{\rho}_{land}\rho Q^{\rho-1}_{land}\\
&P_{nonland} = \lambda\frac{AO_{crop}}{\rho}[(AF_{nonland}Q_{nonland})^{\rho} + (AF_{nonland}Q_{nonland})^{\rho}]^{\frac{1-\rho}{\rho}}AF^{\rho}_{nonland}\rho Q^{\rho-1}_{nonland}\\
&\overline{Q}_{crop} = Q_{crop} = AO_{crop}[(AF_{land}Q_{land})^{\rho} + (AF_{nonland}Q_{nonland})^{\rho}]^{\frac{1}{\rho}},
\end{aligned} \tag{3}$$

while input prices can also be written as

$$\begin{aligned}
&P_{land} = \lambda AO^{\rho}_{crop}AF^{\rho}_{land}Q^{\rho-1}_{land}Q^{1-\rho}_{crop}\\
&P_{nonland} = \lambda AO^{\rho}_{crop}AF^{\rho}_{nonland}Q^{\rho-1}_{nonland}Q^{1-\rho}_{crop}.
\end{aligned} \tag{4}$$

Eliminating λ, we get

$$\frac{P_{land}}{P_{nonland}} = \left(\frac{AF_{land}}{AF_{nonland}}\right)^{\rho}\left(\frac{Q_{land}}{Q_{nonland}}\right)^{\rho-1}. \tag{5}$$

Rearranging (5) to derive quantity ratios

$$\frac{Q_{land}}{Q_{nonland}} = \left(\frac{AF_{land}}{AF_{nonland}}\right)^{\frac{\rho}{1-\rho}}\left(\frac{P_{land}}{P_{nonland}}\right)^{\frac{1}{\rho-1}}, \tag{6}$$

By rearranging the fraction in (6), we have

$$Q_{land} = \left(\frac{AF_{land}}{AF_{nonland}}\right)^{\frac{\rho}{1-\rho}}\left(\frac{P_{land}}{P_{nonland}}\right)^{\frac{1}{\rho-1}} Q_{nonland}. \tag{7}$$

Similarly,

$$Q_{nonland} = \left(\frac{AF_{land}}{AF_{nonland}}\right)^{\frac{\rho}{\rho-1}}\left(\frac{P_{land}}{P_{nonland}}\right)^{\frac{1}{1-\rho}} Q_{land}. \tag{8}$$

Now deriving these conditional factor demands that are exogenous to the other input, by inserting (6) and (7) to farmers' zero profit condition

$$P_{crop}Q_{crop} = P_{land}Q_{land} + P_{nonland}Q_{nonland}. \tag{9}$$

Plugging (4) in (9) gives

$$\begin{aligned} P_{crop}Q_{crop} &= \lambda AO_{crop}^{\rho} AF_{land}^{\rho} Q_{land}^{\rho-1} Q_{crop}^{1-\rho} Q_{land} \\ &+ \lambda AO_{crop}^{\rho} AF_{nonland}^{\rho} Q_{nonland}^{\rho-1} Q_{crop}^{1-\rho} Q_{nonland} \\ &= \lambda Q_{crop}^{1-\rho} AO_{crop}^{\rho} [AF_{land}^{\rho} Q_{land}^{\rho-1} Q_{land} \\ &+ AF_{nonland}^{\rho} Q_{nonland}^{\rho-1} Q_{nonland}] \\ &= \lambda Q_{crop}^{1-\rho} Q_{crop}^{\rho} \\ &= \lambda Q_{crop}, \end{aligned} \tag{10}$$

which gives the aggregate price of crops $P_{crop} = \lambda$.

The optimal input demands are thus given by

$$\begin{aligned} Q^*_{land} &= AO_{crop}^{\frac{\rho}{1-\rho}}(\frac{P_{crop}}{P_{land}})^{\frac{1}{1-\rho}} AF_{land}^{\frac{\rho}{1-\rho}} Q_{crop} \\ Q^*_{nonland} &= AO_{crop}^{\frac{\rho}{1-\rho}}(\frac{P_{crop}}{P_{nonland}})^{\frac{1}{1-\rho}} AF_{nonland}^{\frac{\rho}{1-\rho}} Q_{crop} \end{aligned} \tag{11}$$

Now replacing $\frac{1}{1-\rho}$ with σ, we have

$$\begin{aligned} Q^*_{land} &= (\frac{AO^{\sigma}_{crop}}{AO_{crop}})(\frac{P_{crop}}{P_{land}})^{\sigma}(\frac{AF^{\sigma}_{land}}{AF_{land}})Q_{crop} \\ Q^*_{nonland} &= (\frac{AO^{\sigma}_{crop}}{AO_{crop}})(\frac{P_{crop}}{P_{nonland}})^{\sigma}(\frac{AF^{\sigma}_{nonland}}{AF_{nonland}})Q_{crop}. \end{aligned} \tag{12}$$

Expressing (12) in percentage differences, gives the corresponding linearized equations for *Long Run Derived Demand Equation for Land and Nonland Inputs* in the SIMPLE

$$\begin{aligned} \hat{Q}_{land} + \hat{AF}_{land} &= \hat{Q}_{crop} - \hat{AO}_{crop} - \sigma(\hat{P}_{land} - \hat{AF}_{land} - \hat{P}_{crop} - \hat{AO}_{crop}) \\ \hat{Q}_{nonland} + \hat{AF}_{nonland} &= \hat{Q}_{crop} - \hat{AO}_{crop} - \sigma(\hat{P}_{nonland} - \hat{AF}_{nonland} - \hat{P}_{crop} - \hat{AO}_{crop}). \end{aligned} \tag{13}$$

In TABLO, the two equations in (13) are respectively written as

```
Equation E_QLANDg
    p_QLANDg(g) + p_AFLANDg(g) =p_QCROPg(g) – p_AOCROPg(g)
    – ECROP(g) * [p_PLANDg(g) – pAFLANDg(g) – p_PCROP-p_AOCROPg(g)],
```

and

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Equation E_QNLANDg
 p_QNLANDg(g) + p_AFNLANDg(g) =p_QCROPg(g) – p_AOCROPg(g)
 – ECROP(g) * [p_PNLANDg(g) – p_AFNLANDg(g) – p_PCROP-p_AOCROPg(g)]
```

Now plugging (11) to (9)

$$
\begin{aligned}
P_{crop}Q_{crop} &= P_{land}Q_{land} + P_{nonland}Q_{nonland} \\
&= P_{land}[AO_{crop}^{\frac{\rho}{1-\rho}}(\frac{P_{crop}}{P_{land}})^{\frac{1}{1-\rho}}AF_{land}^{\frac{\rho}{1-\rho}}Q_{crop}] \\
&+ P_{nonland}[AO_{crop}^{\frac{\rho}{1-\rho}}(\frac{P_{crop}}{P_{nonland}})^{\frac{1}{1-\rho}}AF_{nonland}^{\frac{\rho}{1-\rho}}Q_{crop}]
\end{aligned} \tag{14}
$$

Rearranging (14)

$$
\begin{aligned}
P_{crop} &= P_{land}[AO_{crop}^{\frac{\rho}{1-\rho}}(\frac{P_{crop}}{P_{land}})^{\frac{1}{1-\rho}}AF_{land}^{\frac{\rho}{1-\rho}}] \\
&+ P_{nonland}[AO_{crop}^{\frac{\rho}{1-\rho}}(\frac{P_{crop}}{P_{nonland}})^{\frac{1}{1-\rho}}AF_{nonland}^{\frac{\rho}{1-\rho}}] \\
&= \left(AO_{crop}^{\frac{\rho}{1-\rho}}P_{crop}^{\frac{1}{1-\rho}}P_{land}^{\frac{\rho}{\rho-1}}AF_{land}^{\frac{\rho}{1-\rho}}\right) \\
&+ \left(AO_{crop}^{\frac{\rho}{1-\rho}}P_{crop}^{\frac{1}{1-\rho}}P_{nonland}^{\frac{\rho}{\rho-1}}AF_{nonland}^{\frac{\rho}{1-\rho}}\right) \\
&= (AO_{crop}^{\frac{\rho}{1-\rho}}P_{land}^{\frac{\rho}{\rho-1}}AF_{land}^{\frac{\rho}{1-\rho}} + AO_{crop}^{\frac{\rho}{1-\rho}}P_{nonland}^{\frac{\rho}{\rho-1}}AF_{nonland}^{\frac{\rho}{1-\rho}})^{\frac{\rho-1}{\rho}} \\
&= \frac{1}{AO_{crop}}[P_{land}^{\frac{\rho}{\rho-1}}AF_{land}^{\frac{\rho}{1-\rho}} + P_{nonland}^{\frac{\rho}{\rho-1}}AF_{nonland}^{\frac{\rho}{1-\rho}}]^{\frac{\rho-1}{\rho}} \\
&= \frac{1}{AO_{crop}}[P_{land}^{1-\sigma}AF_{land}^{\sigma-1} + P_{nonland}^{1-\sigma}AF_{nonland}^{\sigma-1}]^{\frac{1}{1-\sigma}} \\
\longrightarrow P_{crop}AO_{crop} &= [P_{land}^{1-\sigma}AF_{land}^{\sigma-1} + P_{nonland}^{1-\sigma}AF_{nonland}^{\sigma-1}]^{\frac{1}{1-\sigma}}
\end{aligned} \tag{15}
$$

Applying total differentiation of (15)

$$
\begin{aligned}
&AO_{crop}dP_{crop} + P_{crop}dAO_{crop}\\
&= AO_{crop}P_{crop}(\hat{P}_{crop} + \hat{AO}_{crop})\\
&= \frac{1}{1-\sigma}[P_{land}^{1-\sigma}AF_{land}^{\sigma-1} + P_{nonland}^{1-\sigma}AF_{nonland}^{\sigma-1}]^{\frac{\sigma}{1-\sigma}}\\
&\Big[(\sigma-1)AF_{land}^{\sigma-2}P_{land}^{1-\sigma}dAF_{land} + (\sigma-1)AF_{nonland}^{\sigma-2}P_{nonland}^{1-\sigma}dAF_{nonland}\\
&+ (1-\sigma)P_{land}^{-\sigma}AF_{land}^{\sigma-1}dP_{land} + (1-\sigma)P_{nonland}^{-\sigma}AF_{nonland}^{\sigma-1}dP_{nonland}\Big]\\
&= [P_{land}^{1-\sigma}AF_{land}^{\sigma-1} + P_{nonland}^{1-\sigma}AF_{nonland}^{\sigma-1}]^{\frac{\sigma}{1-\sigma}}\\
&\Big[(-AF_{land}^{\sigma-2}P_{land}^{1-\sigma}dAF_{land} - AF_{nonland}^{\sigma-2}P_{nonland}^{1-\sigma}dAF_{nonland}\\
&+ P_{land}^{-\sigma}AF_{land}^{\sigma-1}dP_{land} + P_{nonland}^{-\sigma}AF_{nonland}^{\sigma-1}dP_{nonland}\Big]\\
&= [P_{land}^{1-\sigma}AF_{land}^{\sigma-1} + P_{nonland}^{1-\sigma}AF_{nonland}^{\sigma-1}]^{\frac{\sigma}{1-\sigma}}\\
&\Big[(-AF_{land}^{\sigma-1}P_{land}^{1-\sigma}\hat{AF}_{land} - AF_{nonland}^{\sigma-1}P_{nonland}^{1-\sigma}\hat{AF}_{nonland}\\
&+ P_{land}^{1-\sigma}AF_{land}^{\sigma-1}\hat{P}_{land} + P_{nonland}^{1-\sigma}AF_{nonland}^{\sigma-1}\hat{P}_{nonland}\Big]\\
&= [P_{land}^{1-\sigma}AF_{land}^{\sigma-1} + P_{nonland}^{1-\sigma}AF_{nonland}^{\sigma-1}]^{\frac{\sigma}{1-\sigma}}\\
&\Big[P_{land}^{1-\sigma}AF_{land}^{\sigma-1}(\hat{P}_{land} - \hat{AF}_{land}) + P_{nonland}^{1-\sigma}AF_{nonland}^{\sigma-1}(\hat{P}_{nonland} - \hat{AF}_{nonland})\Big].
\end{aligned}
\tag{16}
$$

By the expression of $P_{crop}AO_{crop}$ in (15), (16) can be also written as the following

$$
\hat{P}_{crop} + \hat{AO}_{crop} = \pi_{land}(\hat{P}_{land} - \hat{AF}_{land}) + \pi_{nonland}(\hat{P}_{nonland} - \hat{AF}_{nonland}), \tag{17}
$$

where $\pi_{land} = \frac{P_{land}^{1-\sigma}AF_{land}^{\sigma-1}}{P_{land}^{1-\sigma}AF_{land}^{\sigma-1}+P_{nonland}^{1-\sigma}AF_{nonland}^{\sigma-1}}$ and $\pi_{nonland} = \frac{P_{nonland}^{1-\sigma}AF_{nonland}^{\sigma-1}}{P_{land}^{1-\sigma}AF_{land}^{\sigma-1}+P_{nonland}^{1-\sigma}AF_{nonland}^{\sigma-1}}$ are cost shares of land and non-land, respectively, derived from (6). (17) defines *Zero Profit Condition for Crop Producers* in SIMPLE.

4 Modified Land Use in SIMPLE-MAIDADS

4.1 CES Crop Production

In this version, we separate production of crops into oilseeds and non-oil (or other) crop productions. Crops are produced following the same CES technology as in (1), except that each crop has its own production function

$$\begin{cases} Q_{oilcrop} = AO_{crop}[(AF_{oilland}Q_{oilland})^{\frac{\sigma-1}{\sigma}} + (AF_{oilnonland}Q_{oilnonland})^{\frac{\sigma-1}{\sigma}}]^{\frac{\sigma}{\sigma-1}} & \text{(i)} \\ Q_{othercrop} = AO_{crop}[(AF_{otherland}Q_{otherland})^{\frac{\sigma-1}{\sigma}} + (AF_{othernonland}Q_{othernonland})^{\frac{\sigma-1}{\sigma}}]^{\frac{\sigma}{\sigma-1}} & \text{(ii)} \end{cases} \tag{18}$$

Long Run Derived Demand Equations for Lands and Non-Land Inputs

Oilseeds Crops

$$\begin{aligned} &\hat{Q}_{oilland} + \hat{AF}_{oilland} = \hat{Q}_{oilcrop} - \hat{AO}_{crop} - \sigma(\hat{P}_{oilland} - \hat{AF}_{oilland} - \hat{P}_{oilcrop} - \hat{AO}_{crop}) \\ &\hat{Q}_{oilnonland} + \hat{AF}_{oilnonland} = \hat{Q}_{oilcrop} - \hat{AO}_{crop} - \sigma(\hat{P}_{oilnonland} - \hat{AF}_{oilnonland} - \hat{P}_{oilcrop} - \hat{AO}_{crop}) \end{aligned} \tag{19}$$

Other Crops

$$\begin{aligned} &\hat{Q}_{otherland} + \hat{AF}_{otherland} = \hat{Q}_{othercrop} - \hat{AO}_{crop} - \sigma(\hat{P}_{otherland} - \hat{AF}_{otherland} - \hat{P}_{othercrop} - \hat{AO}_{crop}) \\ &\hat{Q}_{othernonland} + \hat{AF}_{othernonland} = \hat{Q}_{othercrop} - \hat{AO}_{crop} - \sigma(\hat{P}_{othernonland} - \hat{AF}_{othernonland} - \hat{P}_{othercrop} \\ &- \hat{AO}_{crop}) \end{aligned} \tag{20}$$

Zero Profit Condition for Crop Producers

Oilseeds Crops

$$\hat{P}_{oilcrop} + \hat{AO}_{crop} = \pi_{oilland}(\hat{P}_{oilland} - \hat{AF}_{oilland}) + \pi_{oilnonland}(\hat{P}_{oilnonland} - \hat{AF}_{oilnonland}) \tag{21}$$

Other Crops

$$\hat{P}_{othercrop} + \hat{AO}_{crop} = \pi_{otherland}(\hat{P}_{otherland} - \hat{AF}_{otherland}) + \pi_{othernonland}(\hat{P}_{othernonland} - \hat{AF}_{othernonland}) \tag{22}$$

where

$$\pi_{oilland} = \frac{P_{oilland}Q_{oilland}}{P_{oilland}Q_{oilland} + P_{oilnonland}Q_{oilnonland}} = \frac{P_{oilland}Q_{oilland}}{P_{oilcrop}Q_{oilcrop}} \tag{23}$$

is the cost share of land inputs used in oilseeds production, and

$$\pi_{oilnonland} = \frac{P_{oilnonland}Q_{oilnonland}}{P_{oilland}Q_{oilland} + P_{oilnonland}Q_{oilnonland}} = \frac{P_{oilnonland}Q_{oilnonland}}{P_{oilcrop}Q_{oilcrop}} \tag{24}$$

is the cost share of non-land inputs in oilseeds production, and

$$\pi_{otherland} = \frac{P_{otherland}Q_{otherland}}{P_{otherland}Q_{otherland} + P_{othernonland}Q_{othernonland}} = \frac{P_{otherland}Q_{otherland}}{P_{othercrop}Q_{othercrop}} \tag{25}$$

is the cost share of land inputs used in other-crop production, and

$$\pi_{otherland} = \frac{P_{othernonland}Q_{othernonland}}{P_{otherland}Q_{otherland} + P_{othernonland}Q_{othernonland}} = \frac{P_{othernonland}Q_{othernonland}}{P_{othercrop}Q_{othercrop}} \tag{26}$$

is the cost share of non-land inputs in other-crop production.

4.2 A Simple CET Allocation of Land Endowment

The CET function below describes competition of different land uses

$$Q_{land} = AO_{land}\big(Q_{oilland}^{\zeta} + Q_{otherland}^{\zeta}\big)^{\frac{1}{\zeta}}, \tag{27}$$

with $\frac{1}{1-\zeta} = \eta$, where $\eta < 0$ is the CET transformation elasticity; AO_{land} is an aggregate shifter of land supply. Farmer maximizes profit subject to (27), which gives the following Lagrangian function

$$\mathcal{L} = P_{oilland}Q_{oilland} + P_{otherland}Q_{otherland} - \Lambda\Big[AO_{land}\big(Q_{oilland}^{\zeta} + Q_{otherland}^{\zeta}\big)^{\frac{1}{\zeta}} - Q_{land}\Big] \tag{28}$$

First-order conditions give

$$P_{oilland} = \Lambda \frac{AO_{land}}{\zeta} \big(Q^{\zeta}_{oilland} + Q^{\zeta}_{otherland} \big)^{\frac{1-\zeta}{\zeta}} \zeta Q^{\zeta-1}_{oilland} \tag{29}$$

$$P_{otherland} = \Lambda \frac{AO_{land}}{\zeta} \big(Q^{\zeta}_{oilland} + Q^{\zeta}_{otherland} \big)^{\frac{1-\zeta}{\zeta}} \zeta Q^{\zeta-1}_{otherland} \tag{30}$$

$$Q_{land} = AO_{land} \big(Q^{\zeta}_{oilland} + Q^{\zeta}_{otherland} \big)^{\frac{1}{\zeta}} \tag{31}$$

with conditions above, we may also write

$$\begin{aligned} P_{oilland} &= \Lambda AO^{\zeta}_{land} Q^{\zeta-1}_{oilland} Q^{1-\zeta}_{land} \\ P_{otherland} &= \Lambda AO^{\zeta}_{land} Q^{\zeta-1}_{otherland} Q^{1-\zeta}_{land} \end{aligned} \tag{32}$$

The optimal quantity supply of lands are thus given by

$$\begin{aligned} Q^{*}_{oilland} &= AO^{\frac{\zeta}{1-\zeta}}_{land} \left(\frac{P_{land}}{P_{otherland}}\right)^{\frac{1}{1-\zeta}} Q_{land} \\ Q^{*}_{otherland} &= AO^{\frac{\zeta}{1-\zeta}}_{land} \left(\frac{P_{land}}{P_{otherland}}\right)^{\frac{1}{1-\zeta}} Q_{land} \end{aligned} \tag{33}$$

Now replacing $\frac{1}{1-\zeta}$ with η

$$\begin{aligned} Q^{*}_{oilland} &= \left(\frac{AO^{\eta}_{land}}{AO_{land}}\right) \left(\frac{P_{land}}{P_{oilland}}\right)^{\eta} Q_{land} \\ Q^{*}_{otherland} &= \left(\frac{AO^{\eta}_{land}}{AO_{land}}\right) \left(\frac{P_{land}}{P_{otherland}}\right)^{\eta} Q_{land} \end{aligned} \tag{34}$$

Expressing (34) in linearzied terms

$$\begin{aligned} \hat{Q}_{oilland} &= \hat{Q}_{land} - \hat{AO}_{land} - \eta(\hat{P}_{oilland} - \hat{P}_{land} - \hat{AO}_{land}) \\ \hat{Q}_{otherland} &= \hat{Q}_{land} - \hat{AO}_{land} - \eta(\hat{P}_{otherland} - \hat{P}_{land} - \hat{AO}_{land}) \end{aligned} \tag{35}$$

Similar to the case derived in the CES crop-production, the zero profit condition for land owners in percentage differences can be expressed as

$$\hat{P}_{land} + \hat{AO}_{land} = S_{oilland}\hat{P}_{oilland} + S_{otherland}\hat{P}_{otherland}, \tag{36}$$

where

$$S_{oilland} = \frac{P_{oilland}Q_{oilland}}{P_{oilland}Q_{oilland} + P_{otherland}Q_{otherland}} = \frac{P_{oilland}Q_{oilland}}{P_{oilcrop}Q_{oilcrop} + P_{othercrop}Q_{othercrop}} \tag{37}$$

is the value share of land planted with oilseeds crops, and

$$S_{otherland} = \frac{P_{otherland}Q_{otherland}}{P_{oilland}Q_{oilland} + P_{otherland}Q_{otherland}} = \frac{P_{otherland}Q_{otherand}}{P_{oilcrop}Q_{oilcrop} + P_{othercrop}Q_{othercrop}} \tag{38}$$

is the value share of land planted with other crops.

Note that another CET tradition is to assume that $\eta = \frac{1}{\zeta-1} > 0$ is the transformation elasticity, which implies that $\zeta = \frac{\eta+1}{\eta}$. It will not affect the linearized zero-profit condition, but will slightly change the expression for the quantity of land supply. Specifically, instead of replacing $\frac{1}{1-\zeta}$ with η in (33), we replace η with $\frac{1}{\zeta-1}$, in which case (35) becomes

$$\begin{aligned} \hat{Q}_{oilland} &= \hat{Q}_{land} - \hat{AO}_{land} + \eta(\hat{P}_{oilland} - \hat{P}_{land} - \hat{AO}_{land}) \\ \hat{Q}_{otherland} &= \hat{Q}_{land} - \hat{AO}_{land} + \eta(\hat{P}_{otherland} - \hat{P}_{land} - \hat{AO}_{land}). \end{aligned} \tag{39}$$

References

[1] Baldos, Uris Lantz C., and Thomas W. Hertel. *SIMPLE: a Simplified International Model of agricultural Prices, Land use and the Environment.* No. 1237-2019-230. 2012.

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[4] Yang, Anton C., Christophe Gouel, and Thomas W. Hertel. "Will Income or Population be the Main Driver of Food Demand Growth to 2050?." (2018).