Optimization with Hat Derivatives: Theoretical and Mathematical Derivation of an Applied Computable Partial Equilibrium Model*

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1 Background

This note links the mathematical concept of **hat derivatives** to constrained optimization results derived and used by economists which is increasingly often written in the General Equilibrium Modelling (GEMPACK) Software using the *TABLO language* of writing economic models.[1] Without these connections, it is fair to say that there are at least two substantial gaps in understanding many simple concept-based but probably lengthy economic models that are almost exclusively implemented using computers today. The first one is that, despite its importance, mathematicians almost never (very rarely, if so) formally introduce hat derivatives in a text book or publicly accessible academic sources (Maurer, 2002); the second one is that, despite its frequent use in contemporary economic computations, economists and economic modelers unwittingly lack a step-by-step explanation of how computer languages use the concept of hat derivatives derived from the neoclassical economic theory. Inside and outside of the economics discipline, this has created barriers to economists' understanding of important model structures and communications among them

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[1]For overview of the TABLO language, see `https://www.http://www.copsmodels.com/gpmanual.htm`.

within their own community. Quality economic research can be disregarded outside of the discipline without the proper and thorough communication of the mathematics. Therefore, the purpose of this note (partially due to increasing demand for relevant inquiries) is to attempt to fill these gaps.

To do so, we use a *Simplified International Model of Agricultural Prices, Land use and the Environment* (SIMPLE) Model (Baldos and Hertel, 2012)–which is an applied computable partial equilibrium model of global agricultural supply and demand–as an example; we show how equations corresponding to land use changes in this model, written in TABLO language, were converted and algebraically derived from producer theory in microeconomics. We then bring a slightly more general case and use a derivative of this model–SIMPLE-MAIDADS (Yang, Gouel and Hertel, 2018) to show derivations with interactions between long-run competition for land and multiple agricultural outputs.[2] Nevertheless, we hope that this note will have much wider adaptability and usefulness, not merely in partial equilibrium models, but as well as in general equilibrium models. For example, the sequence of steps to arrive at the GEMPACK code and mathematical notations are highly similar to those used in the GTAP (Global Trade Analysis Project) Model, which also uses the technique of hat calculus to derive the percentage deviation from the base scenario in economic simulations (e.g., counterfactual analyses carried out on computers). Upon completion, we hope that this note will be beneficial to users who use facilitated interface of simulated economic shocks, while wishing to see the theoretical foundation of its analytical framework.

2 Logarithmic Derivative

Let us denote x' by infinitesimal **absolute change** in an economic variable $x > 0$ (e.g., strictly positive quantity of consumption of fresh apples) where it is also a derivative of x, then $\log' x = x'/x = \hat{x}$ is referred to by mathematicians as infinitesimal **relative change** in x, representing the absolute change in apple consumption relative to (scaled by) the initial quantities of apple consumption, where $\hat{x}$ is the **percentage rate of change** in x or the **instantaneous rate of relative change** to x. Mathematically, we may prove this equivalence using the limit of the average relative rate (Maurer, 2002); denote f by a function of x and that Δf is the absolute change in f, then $\hat{x} = \lim_{\Delta f \to 0} \frac{\Delta x / \Delta f}{x} = \frac{x'}{x} = \log' x$, where $\Delta x = x(f + \Delta f) - x(f)$ is the absolute change in x. Therefore, the defined redux from the concept used in calculus (or in functions of complex variables) shows that the term of percentage change (from the base scenario) frequently used by economists is no different from what is widely understood by mathematicians as logarithmic derivative.

Rules of the Percent Rate of Change Here we demonstrate some important extensions of formulae of hat derivatives used in economic models.[3]

[2] MAIDADS stands for a A Modified, Implicit Directly Additive Demand System (Preckel et al., 2010).

[3] The author thanks to and benefits from Bill Gibson who summarized basic formulae of hat calculus.

The first one is the **(i) rule of multiplication**. If we define that

$$z = xy,$$

then

$$\begin{aligned}\log z &= \log(xy)\\ &= \log x + \log y,\end{aligned}$$

and therefore

$$\begin{aligned}\log' z &= \log'(xy)\\ &= \log' x + \log' y,\end{aligned}$$

which is equivalent as stating that

$$\frac{z'}{z} = \frac{x'}{x} + \frac{y'}{y},$$

or[4]

$$\hat{z} = \hat{x} + \hat{y}.$$

Certainly, if we define c as a constant and replace it with the variable x (that is, if $z = cy$), then by the **(ii) rule of constant multiplication**, we have

$$\hat{z} = \hat{y}.$$

Following the same logic, we may write **(iii) the rule of division**. If we have that

$$z = \frac{x}{y},$$

then

$$\hat{z} = \hat{x} - \hat{y}.$$

Finally, the **(iv) rule of exponents** states that, if $z = x^b$, then

$$\hat{z} = b\hat{x}.$$

[4] Same results can be derived using the product rule, e.g., $\hat{z} = \widehat{xy} = \hat{x} + \hat{y}$.

3 Economics: Land Use in the Standard SIMPLE

We now apply hat derivatives and its extensions and rules in practical economic problems. To benefit users who read TABLO codes and/or use *AnalyseGE* to evaluate and decompose changes in endogenous variables based on changes in exogenous shocks, we adopt variable names that are close to what have been written in the TABLO file.[5] We illustrate how the land use (e.g., farmers' problem) is modeled and solved in the SIMPLE Model and how its expressions in percentage differences, written in TABLO, are derived.[6]

Farmers' constant-elasticity-of-substitution (CES) production technology in the standard SIMPLE takes the form

$$Q_{crop} = AO_{crop}[(AF_{land}Q_{land})^{\rho} + (AF_{nonland}Q_{nonland})^{\rho}]^{\frac{1}{\rho}}. \tag{1}$$

where AO_{crop} is the Hicks-neutral efficiency index (or Total Factor Productivity) in crop production; AF_{land} is the land-biased efficiency index in crop production; $AF_{nonland}$ is the non-land input biased efficiency index in crop production; Q_{crop} is the output quantity of crops; Q_{land} and $Q_{nonland}$ are quantities of land and non-land inputs, respectively; ρ is a CES substitution parameter with $\frac{1}{1-\rho}$ equaling σ.

Farmers minimize expenditure and are subject to the technology constraint

$$\mathcal{L} = P_{land}Q_{land} + P_{nonland}Q_{nonland} + \lambda[\overline{Q}_{crop} - Q_{crop}], \tag{2}$$

where $\overline{Q}_{crop}$ is the targeted output in the production/technology capacity constraint.

First-Order Conditions (F.O.C.) give that

$$\begin{aligned}
&P_{land} = \lambda\frac{AO_{crop}}{\rho}[(AF_{land}Q_{land})^{\rho} + (AF_{nonland}Q_{nonland})^{\rho}]^{\frac{1-\rho}{\rho}}AF_{land}^{\rho}\rho Q_{land}^{\rho-1} \\
&P_{nonland} = \lambda\frac{AO_{crop}}{\rho}[(AF_{nonland}Q_{nonland})^{\rho} + (AF_{nonland}Q_{nonland})^{\rho}]^{\frac{1-\rho}{\rho}}AF_{nonland}^{\rho}\rho Q_{nonland}^{\rho-1} \\
&\overline{Q}_{crop} = Q_{crop} = AO_{crop}[(AF_{land}Q_{land})^{\rho} + (AF_{nonland}Q_{nonland})^{\rho}]^{\frac{1}{\rho}},
\end{aligned} \tag{3}$$

[5] AnalyseGE is an analytical tool of simulation results used in partial or general equilibrium models, e.g., SIMPLE and GTAP models; see also `https://www.copsmodels.com/gpange.htm`.

[6] Source code is available upon request.

while input prices can also be written as

$$
\begin{aligned}
P_{land} &= \lambda AO_{crop}^{\rho} AF_{land}^{\rho} Q_{land}^{\rho-1} Q_{crop}^{1-\rho} \\
P_{nonland} &= \lambda AO_{crop}^{\rho} AF_{nonland}^{\rho} Q_{nonland}^{\rho-1} Q_{crop}^{1-\rho}.
\end{aligned} \tag{4}
$$

Eliminating λ, we get

$$
\frac{P_{land}}{P_{nonland}} = \left(\frac{AF_{land}}{AF_{nonland}}\right)^{\rho} \left(\frac{Q_{land}}{Q_{nonland}}\right)^{\rho-1}. \tag{5}
$$

Rearranging (5) to derive quantity ratios

$$
\frac{Q_{land}}{Q_{nonland}} = \left(\frac{AF_{land}}{AF_{nonland}}\right)^{\frac{\rho}{1-\rho}} \left(\frac{P_{land}}{P_{nonland}}\right)^{\frac{1}{\rho-1}}. \tag{6}
$$

By rearranging the fraction in (6), we have

$$
Q_{land} = \left(\frac{AF_{land}}{AF_{nonland}}\right)^{\frac{\rho}{1-\rho}} \left(\frac{P_{land}}{P_{nonland}}\right)^{\frac{1}{\rho-1}} Q_{nonland}. \tag{7}
$$

Similarly,

$$
Q_{nonland} = \left(\frac{AF_{land}}{AF_{nonland}}\right)^{\frac{\rho}{\rho-1}} \left(\frac{P_{land}}{P_{nonland}}\right)^{\frac{1}{1-\rho}} Q_{land}. \tag{8}
$$

Now deriving these conditional factor demands that are exogenous to the other input, by inserting (7) and (8) to farmers' zero profit condition

$$
P_{crop} Q_{crop} = P_{land} Q_{land} + P_{nonland} Q_{nonland}. \tag{9}
$$

Plugging (4) in (9) gives

$$
\begin{aligned}
P_{crop}Q_{crop} &= \lambda AO^{\rho}_{crop}AF^{\rho}_{land}Q^{\rho-1}_{land}Q^{1-\rho}_{crop}Q_{land} \\
&+ \lambda AO^{\rho}_{crop}AF^{\rho}_{nonland}Q^{\rho-1}_{nonland}Q^{1-\rho}_{crop}Q_{nonland} \\
&= \lambda Q^{1-\rho}_{crop}AO^{\rho}_{crop}[AF^{\rho}_{land}Q^{\rho-1}_{land}Q_{land} \\
&+ AF^{\rho}_{nonland}Q^{\rho-1}_{nonland}Q_{nonland}] \\
&= \lambda Q^{1-\rho}_{crop}Q^{\rho}_{crop} \\
&= \lambda Q_{crop},
\end{aligned} \tag{10}
$$

which gives that the aggregate price of crops $P_{crop} = \lambda$.

Substituting λ with P_{crop} in Eq. (4), the optimal input demands are thus given by

$$
\begin{aligned}
Q^*_{land} &= AO^{\frac{\rho}{1-\rho}}_{crop}(\frac{P_{crop}}{P_{land}})^{\frac{1}{1-\rho}}AF^{\frac{\rho}{1-\rho}}_{land}Q_{crop} \\
Q^*_{nonland} &= AO^{\frac{\rho}{1-\rho}}_{crop}(\frac{P_{crop}}{P_{nonland}})^{\frac{1}{1-\rho}}AF^{\frac{\rho}{1-\rho}}_{nonland}Q_{crop}.
\end{aligned} \tag{11}
$$

Now replacing $\frac{1}{1-\rho}$ with σ, we have

$$
\begin{aligned}
Q^*_{land} &= (\frac{AO^{\sigma}_{crop}}{AO_{crop}})(\frac{P_{crop}}{P_{land}})^{\sigma}(\frac{AF^{\sigma}_{land}}{AF_{land}})Q_{crop} \\
Q^*_{nonland} &= (\frac{AO^{\sigma}_{crop}}{AO_{crop}})(\frac{P_{crop}}{P_{nonland}})^{\sigma}(\frac{AF^{\sigma}_{nonland}}{AF_{nonland}})Q_{crop}.
\end{aligned} \tag{12}
$$

Using the **rules of the percent rate of change** to express (12) in percentage differences, it gives the corresponding linearized equations for *Long Run Derived Demand Equation for Land and Nonland Inputs* in the SIMPLE Model

$$
\begin{aligned}
&\hat{Q}_{land} + \hat{AF}_{land} = \hat{Q}_{crop} - \hat{AO}_{crop} - \sigma(\hat{P}_{land} - \hat{AF}_{land} - \hat{P}_{crop} - \hat{AO}_{crop}) \\
&\hat{Q}_{nonland} + \hat{AF}_{nonland} = \hat{Q}_{crop} - \hat{AO}_{crop} - \sigma(\hat{P}_{nonland} - \hat{AF}_{nonland} - \hat{P}_{crop} - \hat{AO}_{crop}).
\end{aligned} \tag{13}
$$

In TABLO, the two equations in (13) are respectively written as

Equation E_QLANDg
p_QLANDg(g) + p_AFLANDg(g) =p_QCROPg(g) – p_AOCROPg(g)
– ECROP(g) * [p_PLANDg(g) – pAFLANDg(g) – p_PCROP-p_AOCROPg(g)],

and

Equation E_QNLANDg
p_QNLANDg(g) + p_AFNLANDg(g) =p_QCROPg(g) – p_AOCROPg(g)
– ECROP(g) * [p_PNLANDg(g) – p_AFNLANDg(g) – p_PCROP-p_AOCROPg(g)]

Now plugging (11) to (9)

$$\begin{aligned} P_{crop}Q_{crop} &= P_{land}Q_{land} + P_{nonland}Q_{nonland} \\ &= P_{land}[AO_{crop}^{\frac{\rho}{1-\rho}}(\frac{P_{crop}}{P_{land}})^{\frac{1}{1-\rho}}AF_{land}^{\frac{\rho}{1-\rho}}Q_{crop}] \\ &+ P_{nonland}[AO_{crop}^{\frac{\rho}{1-\rho}}(\frac{P_{crop}}{P_{nonland}})^{\frac{1}{1-\rho}}AF_{nonland}^{\frac{\rho}{1-\rho}}Q_{crop}] \end{aligned} \tag{14}$$

Rearranging (14)

$$\begin{aligned} P_{crop} &= P_{land}[AO_{crop}^{\frac{\rho}{1-\rho}}(\frac{P_{crop}}{P_{land}})^{\frac{1}{1-\rho}}AF_{land}^{\frac{\rho}{1-\rho}}] \\ &+ P_{nonland}[AO_{crop}^{\frac{\rho}{1-\rho}}(\frac{P_{crop}}{P_{nonland}})^{\frac{1}{1-\rho}}AF_{nonland}^{\frac{\rho}{1-\rho}}] \\ &= \left(AO_{crop}^{\frac{\rho}{1-\rho}}P_{crop}^{\frac{1}{1-\rho}}P_{land}^{\frac{\rho}{\rho-1}}AF_{land}^{\frac{\rho}{1-\rho}}\right) \\ &+ \left(AO_{crop}^{\frac{\rho}{1-\rho}}P_{crop}^{\frac{1}{1-\rho}}P_{nonland}^{\frac{\rho}{\rho-1}}AF_{nonland}^{\frac{\rho}{1-\rho}}\right) \\ &= (AO_{crop}^{\frac{\rho}{1-\rho}}P_{land}^{\frac{\rho}{\rho-1}}AF_{land}^{\frac{\rho}{1-\rho}} + AO_{crop}^{\frac{\rho}{1-\rho}}P_{nonland}^{\frac{\rho}{\rho-1}}AF_{nonland}^{\frac{\rho}{1-\rho}})^{\frac{\rho-1}{\rho}} \\ &= \frac{1}{AO_{crop}}[P_{land}^{\frac{\rho}{\rho-1}}AF_{land}^{\frac{\rho}{1-\rho}} + P_{nonland}^{\frac{\rho}{\rho-1}}AF_{nonland}^{\frac{\rho}{1-\rho}}]^{\frac{\rho-1}{\rho}} \\ &= \frac{1}{AO_{crop}}[P_{land}^{1-\sigma}AF_{land}^{\sigma-1} + P_{nonland}^{1-\sigma}AF_{nonland}^{\sigma-1}]^{\frac{1}{1-\sigma}} \\ \longrightarrow P_{crop}AO_{crop} &= [P_{land}^{1-\sigma}AF_{land}^{\sigma-1} + P_{nonland}^{1-\sigma}AF_{nonland}^{\sigma-1}]^{\frac{1}{1-\sigma}} \end{aligned} \tag{15}$$

Applying total differentiation of (15)

$$
\begin{aligned}
&AO_{crop}dP_{crop} + P_{crop}dAO_{crop} \\
&= AO_{crop}P_{crop}(\hat{P}_{crop} + \hat{AO}_{crop}) \\
&= \frac{1}{1-\sigma}[P_{land}^{1-\sigma}AF_{land}^{\sigma-1} + P_{nonland}^{1-\sigma}AF_{nonland}^{\sigma-1}]^{\frac{\sigma}{1-\sigma}} \\
&\bigg[(\sigma-1)AF_{land}^{\sigma-2}P_{land}^{1-\sigma}dAF_{land} + (\sigma-1)AF_{nonland}^{\sigma-2}P_{nonland}^{1-\sigma}dAF_{nonland} \\
&+ (1-\sigma)P_{land}^{-\sigma}AF_{land}^{\sigma-1}dP_{land} + (1-\sigma)P_{nonland}^{-\sigma}AF_{nonland}^{\sigma-1}dP_{nonland}\bigg] \\
&= [P_{land}^{1-\sigma}AF_{land}^{\sigma-1} + P_{nonland}^{1-\sigma}AF_{nonland}^{\sigma-1}]^{\frac{\sigma}{1-\sigma}} \\
&\bigg[(-AF_{land}^{\sigma-2}P_{land}^{1-\sigma}dAF_{land} - AF_{nonland}^{\sigma-2}P_{nonland}^{1-\sigma}dAF_{nonland} \\
&+ P_{land}^{-\sigma}AF_{land}^{\sigma-1}dP_{land} + P_{nonland}^{-\sigma}AF_{nonland}^{\sigma-1}dP_{nonland}\bigg] \\
&= [P_{land}^{1-\sigma}AF_{land}^{\sigma-1} + P_{nonland}^{1-\sigma}AF_{nonland}^{\sigma-1}]^{\frac{\sigma}{1-\sigma}} \\
&\bigg[(-AF_{land}^{\sigma-1}P_{land}^{1-\sigma}\hat{AF}_{land} - AF_{nonland}^{\sigma-1}P_{nonland}^{1-\sigma}\hat{AF}_{nonland} \\
&+ P_{land}^{1-\sigma}AF_{land}^{\sigma-1}\hat{P}_{land} + P_{nonland}^{1-\sigma}AF_{nonland}^{\sigma-1}\hat{P}_{nonland}\bigg] \\
&= [P_{land}^{1-\sigma}AF_{land}^{\sigma-1} + P_{nonland}^{1-\sigma}AF_{nonland}^{\sigma-1}]^{\frac{\sigma}{1-\sigma}} \\
&\bigg[P_{land}^{1-\sigma}AF_{land}^{\sigma-1}(\hat{P}_{land} - \hat{AF}_{land}) + P_{nonland}^{1-\sigma}AF_{nonland}^{\sigma-1}(\hat{P}_{nonland} - \hat{AF}_{nonland})\bigg].
\end{aligned}
\tag{16}
$$

By the expression of $P_{crop}AO_{crop}$ in (15), (16) can be also written as the following

$$
\hat{P}_{crop} + \hat{AO}_{crop} = \pi_{land}(\hat{P}_{land} - \hat{AF}_{land}) + \pi_{nonland}(\hat{P}_{nonland} - \hat{AF}_{nonland}), \tag{17}
$$

where $\pi_{land} = \frac{P_{land}^{1-\sigma}AF_{land}^{\sigma-1}}{P_{land}^{1-\sigma}AF_{land}^{\sigma-1} + P_{nonland}^{1-\sigma}AF_{nonland}^{\sigma-1}}$ and $\pi_{nonland} = \frac{P_{nonland}^{1-\sigma}AF_{nonland}^{\sigma-1}}{P_{land}^{1-\sigma}AF_{land}^{\sigma-1} + P_{nonland}^{1-\sigma}AF_{nonland}^{\sigma-1}}$ are cost shares of land and non-land, respectively, derived from (6). Eq. (17) defines the *Zero Profit Condition for Crop Producers* in the SIMPLE Model.

4 Modified Land Use in SIMPLE-MAIDADS

4.1 CES Crop Production

In this version, we separate production of crops into oilseeds and non-oil (or other) crop productions. Crops are produced following the same CES technology as in (1), except that each crop has its own production function

$$\begin{cases} Q_{oilcrop} = AO_{crop}[(AF_{oilland}Q_{oilland})^{\frac{\sigma-1}{\sigma}} + (AF_{oilnonland}Q_{oilnonland})^{\frac{\sigma-1}{\sigma}}]^{\frac{\sigma}{\sigma-1}} & \text{(i)} \\ Q_{othercrop} = AO_{crop}[(AF_{otherland}Q_{otherland})^{\frac{\sigma-1}{\sigma}} + (AF_{othernonland}Q_{othernonland})^{\frac{\sigma-1}{\sigma}}]^{\frac{\sigma}{\sigma-1}} & \text{(ii)} \end{cases} \tag{18}$$

Long Run Derived Demand Equations for Lands and Non-Land Inputs

Oilseeds Crops

$$\begin{aligned} \hat{Q}_{oilland} + \hat{AF}_{oilland} &= \hat{Q}_{oilcrop} - \hat{AO}_{crop} - \sigma(\hat{P}_{oilland} - \hat{AF}_{oilland} - \hat{P}_{oilcrop} - \hat{AO}_{crop}) \\ \hat{Q}_{oilnonland} + \hat{AF}_{oilnonland} &= \hat{Q}_{oilcrop} - \hat{AO}_{crop} - \sigma(\hat{P}_{oilnonland} - \hat{AF}_{oilnonland} - \hat{P}_{oilcrop} - \hat{AO}_{crop}) \end{aligned} \tag{19}$$

Other Crops

$$\begin{aligned} \hat{Q}_{otherland} + \hat{AF}_{otherland} &= \hat{Q}_{othercrop} - \hat{AO}_{crop} - \sigma(\hat{P}_{otherland} - \hat{AF}_{otherland} - \hat{P}_{othercrop} - \hat{AO}_{crop}) \\ \hat{Q}_{othernonland} + \hat{AF}_{othernonland} &= \hat{Q}_{othercrop} - \hat{AO}_{crop} - \sigma(\hat{P}_{othernonland} - \hat{AF}_{othernonland} - \hat{P}_{othercrop} \\ &\quad - \hat{AO}_{crop}) \end{aligned} \tag{20}$$

Zero Profit Condition for Crop Producers

Oilseeds Crops

$$\hat{P}_{oilcrop} + \hat{AO}_{crop} = \pi_{oilland}(\hat{P}_{oilland} - \hat{AF}_{oilland}) + \pi_{oilnonland}(\hat{P}_{oilnonland} - \hat{AF}_{oilnonland}) \tag{21}$$

Other Crops

$$\hat{P}_{othercrop} + \hat{AO}_{crop} = \pi_{otherland}(\hat{P}_{otherland} - \hat{AF}_{otherland}) + \pi_{othernonland}(\hat{P}_{othernonland} - \hat{AF}_{othernonland}) \tag{22}$$

where

$$\pi_{oilland} = \frac{P_{oilland}Q_{oilland}}{P_{oilland}Q_{oilland} + P_{oilnonland}Q_{oilnonland}} = \frac{P_{oilland}Q_{oilland}}{P_{oilcrop}Q_{oilcrop}} \tag{23}$$

is the cost share of land inputs used in oilseeds production, and

$$\pi_{oilnonland} = \frac{P_{oilnonland}Q_{oilnonland}}{P_{oilland}Q_{oilland} + P_{oilnonland}Q_{oilnonland}} = \frac{P_{oilnonland}Q_{oilnonland}}{P_{oilcrop}Q_{oilcrop}} \tag{24}$$

is the cost share of non-land inputs in oilseeds production, and

$$\pi_{otherland} = \frac{P_{otherland}Q_{otherland}}{P_{otherland}Q_{otherland} + P_{othernonland}Q_{othernonland}} = \frac{P_{otherland}Q_{otherland}}{P_{othercrop}Q_{othercrop}} \tag{25}$$

is the cost share of land inputs used in other-crop production, and

$$\pi_{otherland} = \frac{P_{othernonland}Q_{othernonland}}{P_{otherland}Q_{otherland} + P_{othernonland}Q_{othernonland}} = \frac{P_{othernonland}Q_{othernonland}}{P_{othercrop}Q_{othercrop}} \tag{26}$$

is the cost share of non-land inputs in other-crop production.

4.2 A Simple CET Allocation of Land Endowment

The constant-elasticity-of-transformation (CET) function below describes competition of different land uses

$$Q_{land} = AO_{land}\big(Q_{oilland}^{\zeta} + Q_{otherland}^{\zeta}\big)^{\frac{1}{\zeta}}, \tag{27}$$

with $\frac{1}{1-\zeta} = \eta$, where $\eta < 0$ is the CET transformation elasticity; AO_{land} is an aggregate shifter of land supply. Farmer maximizes profit subject to (27), which gives the following Lagrangian function

$$\mathcal{L} = P_{oilland}Q_{oilland} + P_{otherland}Q_{otherland} - \Lambda\Big[AO_{land}\big(Q_{oilland}^{\zeta} + Q_{otherland}^{\zeta}\big)^{\frac{1}{\zeta}} - Q_{land}\Big] \tag{28}$$

F.O.C. give

$$P_{oilland} = \Lambda \frac{AO_{land}}{\zeta} \big(Q_{oilland}^{\zeta} + Q_{otherland}^{\zeta}\big)^{\frac{1-\zeta}{\zeta}} \zeta Q_{oilland}^{\zeta-1} \tag{29}$$

$$P_{otherland} = \Lambda \frac{AO_{land}}{\zeta} \big(Q_{oilland}^{\zeta} + Q_{otherland}^{\zeta}\big)^{\frac{1-\zeta}{\zeta}} \zeta Q_{otherland}^{\zeta-1} \tag{30}$$

$$Q_{land} = AO_{land} \big(Q_{oilland}^{\zeta} + Q_{otherland}^{\zeta}\big)^{\frac{1}{\zeta}} \tag{31}$$

with conditions above, we may also write

$$\begin{aligned} P_{oilland} &= \Lambda AO_{land}^{\zeta} Q_{oilland}^{\zeta-1} Q_{land}^{1-\zeta} \\ P_{otherland} &= \Lambda AO_{land}^{\zeta} Q_{otherland}^{\zeta-1} Q_{land}^{1-\zeta} \end{aligned} \tag{32}$$

The optimal quantity supply of lands are thus given by

$$\begin{aligned} Q_{oilland}^{*} &= AO_{land}^{\frac{\zeta}{1-\zeta}} \left(\frac{P_{land}}{P_{otherland}}\right)^{\frac{1}{1-\zeta}} Q_{land} \\ Q_{otherland}^{*} &= AO_{land}^{\frac{\zeta}{1-\zeta}} \left(\frac{P_{land}}{P_{otherland}}\right)^{\frac{1}{1-\zeta}} Q_{land} \end{aligned} \tag{33}$$

Now replacing $\frac{1}{1-\zeta}$ with η

$$\begin{aligned} Q_{oilland}^{*} &= \left(\frac{AO_{land}^{\eta}}{AO_{land}}\right)\left(\frac{P_{land}}{P_{oilland}}\right)^{\eta} Q_{land} \\ Q_{otherland}^{*} &= \left(\frac{AO_{land}^{\eta}}{AO_{land}}\right)\left(\frac{P_{land}}{P_{otherland}}\right)^{\eta} Q_{land} \end{aligned} \tag{34}$$

Expressing (34) in linearzied terms

$$\begin{aligned} \hat{Q}_{oilland} &= \hat{Q}_{land} - \hat{AO}_{land} - \eta(\hat{P}_{oilland} - \hat{P}_{land} - \hat{AO}_{land}) \\ \hat{Q}_{otherland} &= \hat{Q}_{land} - \hat{AO}_{land} - \eta(\hat{P}_{otherland} - \hat{P}_{land} - \hat{AO}_{land}) \end{aligned} \tag{35}$$

Similar to the case derived in the CES crop-production, the zero profit condition for land owners in percentage differences can be expressed as

$$\hat{P}_{land} + \hat{AO}_{land} = S_{oilland}\hat{P}_{oilland} + S_{otherland}\hat{P}_{otherland}, \tag{36}$$

where

$$S_{oilland} = \frac{P_{oilland}Q_{oilland}}{P_{oilland}Q_{oilland} + P_{otherland}Q_{otherland}} = \frac{P_{oilland}Q_{oilland}}{P_{oilcrop}Q_{oilcrop} + P_{othercrop}Q_{othercrop}} \tag{37}$$

is the value share of land planted with oilseeds crops, and

$$S_{otherland} = \frac{P_{otherland}Q_{otherland}}{P_{oilland}Q_{oilland} + P_{otherland}Q_{otherland}} = \frac{P_{otherland}Q_{otherand}}{P_{oilcrop}Q_{oilcrop} + P_{othercrop}Q_{othercrop}} \tag{38}$$

is the value share of land planted with other crops.

Note that another CET tradition is to assume that $\eta = \frac{1}{\zeta-1} > 0$ is the transformation elasticity, which implies that $\zeta = \frac{\eta+1}{\eta}$. It will not affect the linearized zero-profit condition, but will slightly change the expression for the quantity of land supply. Specifically, instead of replacing $\frac{1}{1-\zeta}$ with η in (33), we replace η with $\frac{1}{\zeta-1}$, in which case (35) becomes

$$\begin{aligned} \hat{Q}_{oilland} &= \hat{Q}_{land} - \hat{AO}_{land} + \eta(\hat{P}_{oilland} - \hat{P}_{land} - \hat{AO}_{land}) \\ \hat{Q}_{otherland} &= \hat{Q}_{land} - \hat{AO}_{land} + \eta(\hat{P}_{otherland} - \hat{P}_{land} - \hat{AO}_{land}). \end{aligned} \tag{39}$$

5 Discussions

Computable partial or general equilibrium models have been very powerful analytical models in economics to evaluate various responses of an economy often to exogenous shocks (e.g., SIMPLE and GTAP models). Many of these models are written by economic modelers in a GEMPACK program called TABLO, while economic researchers use an analytical tool–namely AnalyseGE–to analyze the estimated simulation results. The purpose of formulating and implementing these models via micro-theoretical foundations and GEMPACK programs are conceptually similar to counterfactual analyses of other "succinct" or middle-sized economic models with fewer parameters and lines of code (e.g., new quantitative trade models; see also Costinot and Rodríguez-Clare's Handbook of International Economics). However, sometimes they can be quickly difficult to track due to the large size of these models, especially without proper and entirely detailed modeling documentation. Apparently, important economic research involving thousands of lines of code can be valued lightly if we do not give comprehensive explanations of where they come from. This note hopefully adds a little transparency to such existing models by displaying a tighter linkage between micro-theoretical foundations and their derived statement or syntax written in the computer program.

The bottom line is that economists should in general communicate and transmit messages clearly and more deeply to avoid gaps that may cause disconnections among sub-fields and intra-fields of economics. To this end, I would welcome further discussion and debate as to the usefulness of this document and whether and how it should be spread out over time. If you find errors of any types, please **e-mail** me and I will correct them as soon as possible, besides writing a thank-you note, wholeheartedly.

References

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[4] Preckel, Paul V., J.A.L. Cranfield, and Thomas W. Hertel. A Modified, Implicit, Directly Additive Demand System., *Applied Economics*, 42(2), 143–155, 2010.

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