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UNIVERSAL CES DEMAND SYSTEMS AND COUNTERFACTUALS IN INTERNATIONAL TRADE[1]

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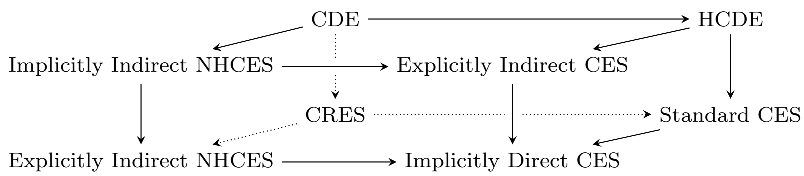
This paper revisits standard and specialized Constant elasticity of substitution (CES) demand systems which we call a *Universal CES Demand System*. We demonstrate, as clear as possible, the transformation of the can-be-nested additive demand systems within this universal parameterized framework illustrated as follows:



Of a particular note is how these transformations can be related to counterfactual analyses in a class of generalizations in quantitative general equilibrium trade models, whether or not they belong to the family of succinct theory-consistent reduced-forms or more sophisticated computational framework (with a well-detailed description of the world economy). Through processes of parameterization, it appears that (1) implicitly additive demand systems are generally *less responsive* to income changes on trade flows and consumer welfare; and (2) indirectly or directly separable demands (on the indifference surface of consumers' quantity choices), whether implicit or explicit, does not draw any distinctions in terms of counterfactuals.

Keywords: CES demand, Counterfactual Changes, Quantitative trade models.

1. CONSTANT DIFFERENCE OF ELASTICITIES (CDE)

The standard CDE is a class of implicitly indirect additive demand system. It is a general case of the standard CES demand system. The nature of the use of the terminology "implicit" rather than "explicit" is that utility in the model cannot be explicitly and algebraically solved using the model's exogenous variables and model parameters. The distinction between "indirect" and "direct" is that indirect additive models are separable in the n unit-cost or normalized prices along consumers' indifference surfaces, whereas directly separable models are additive in n consumer goods (Hanoch (1975)). The standard CDE model is implicitly and indirectly defined as:

$$G(\frac{\mathbf{p}}{w}, u) = \sum_i \beta_i u^{e_i(1-\alpha_i)}(\frac{p_i}{w})^{1-\alpha_i} \equiv 1 \quad \text{(CDE)}. \tag{1.1}$$

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[1]This material is written for the purposes of discussions, idea exchanges, and for the convenience of applying parametric transformations directly in quantitative international trade models. All errors are my own.

Parametric restrictions $\forall i$ for (1.1) to be globally valid (monotonic and quasi-concave), $\forall$ normalized price vectors (e.g., unit-cost prices) $\boldsymbol{\xi} = \frac{\mathbf{p}}{w} \gg \mathbf{0}$, are as follows Hanoch (1975):[1]

$$\begin{cases} \beta_i > 0 & \text{(i)} \\ e_i > 0 & \text{(ii)} \\ \alpha_i > 0 & \text{(iii)} \\ \alpha_i \geq 1 \quad \text{or} \quad 0 < \alpha_i < 1 & \text{(iv)} \end{cases}$$

Using Roy's Identity, the Marshallian or ordinary demand correspondence is given by:

$$q_i(\mathbf{p}, w) = \frac{\left[\beta_i v^{e_i(1-\alpha_i)}(1-\alpha_i)(\frac{p_i}{w})^{-\alpha_i}\right]}{\sum_j \beta_j v^{e_j(1-\alpha_j)}(1-\alpha_j)(\frac{p_j}{w})^{1-\alpha_j}}. \tag{1.2}$$

2. IMPLICITLY INDIRECT NON-HOMOTHETIC CES (NHCES)

PROPOSITION 1 *Let $G(\boldsymbol{\xi}, u)$ be an implicit indirect utility function of Constant Difference of Elasticities (CDE), then $G(\boldsymbol{\xi}, u)$ can be parameterized to be an implicit indirect version of non-homothetic CES function, which is identical to implicit direct non-homothetic CES.*

PROOF: Let $\alpha_i = \alpha \ \forall i$, then (1.1) generalizes to:

$$G(\frac{\mathbf{p}}{w}, u) = \sum_i \beta_i u^{e_i(1-\alpha)}(\frac{p_i}{w})^{1-\alpha} \equiv 1 \quad \text{(Implicitly Indirect NHCES)}. \tag{2.1}$$

(1) *Constant elasticity of substitution* The cross-Allen partial elasticity is:

$$\begin{aligned} \sigma_{ij} &= \alpha_i + \alpha_j - \sum_k \pi_k \alpha_k - \frac{\Delta_{ij}\alpha_i}{\pi_i} \\ &= 2\alpha - \alpha \sum_k \pi_k - \frac{\Delta_{ij}\alpha}{\pi_i} \qquad \Longleftrightarrow \quad \alpha_i = \alpha \quad \forall i \\ &= \alpha\left(2 - \sum_k \pi_k - \frac{\Delta_{ij}}{\pi_i}\right) \\ &= \alpha\left(1 - \frac{\Delta_{ij}}{\pi_i}\right) \\ &= \alpha \quad \forall i \neq j. \end{aligned} \tag{2.2}$$

[1] Condition (iv) automatically satisfies (iii), but they are not equivalently meaningful for the global or local regularity condition. The model is *dampened* to be locally valid if the "same sign" condition in (iv) is not satisfied, and $\alpha_I < 0$ for some $I \in i$, which violates (iii).

(2) *Identical to implicit direct NHCES* By §2.2 and §2.4 in Hanoch (1975):[2]
Taking the natural logarithm of both sides of (1.2) (with $\xi_i = p_i/w$):

$$\ln q_i = \ln[\beta_i(1-\alpha_i)] + e_i(1-\alpha_i)\ln u - \alpha_i \ln \xi_i - \ln\Big[\sum_j \beta_j u^{e_j(1-\alpha_j)}(1-\alpha_j)\xi_i^{1-\alpha_j}\Big]. \tag{2.3}$$

Eliminating the last term in equation (2.3) by using logarithmic ratio:

$$\begin{aligned}\ln\frac{q_i}{q_1} &= \ln\frac{\beta_i(1-\alpha_i)}{\beta_1(1-\alpha_1)} + [e_i(1-\alpha_i) - e_1(1-\alpha_1)]\ln u - \alpha_i\ln\xi_i + \alpha_1\ln\xi_1 \\ &= A_i + Z_i\ln u - \alpha_i\ln\xi_i + \alpha_1\ln\xi_1 \quad \forall i \in [2,\infty) \\ &= \widetilde{A}_i + \widetilde{Z}_i\ln u - \alpha\ln(\frac{p_i}{p_1}) \quad \forall i\in[2,\infty) \quad \iff \quad \alpha_i = \alpha \quad \forall i,\end{aligned} \tag{2.4}$$

where $A_i = \ln\frac{\beta_i(1-\alpha_i)}{\beta_1(1-\alpha_1)}$, $Z_i = e_i(1-\alpha_i) - e_1(1-\alpha_1)$; $\widetilde{A}_i = \ln\frac{\beta_i}{\beta_1}$, $\widetilde{Z}_i = (e_i - e_1)(1-\alpha)$.

Considering the following implicitly direct NHCES function in Hanoch (1975):

$$F(\mathbf{q}, u) = \sum_i k_i u^{-e_i(1-g)} q_i^{1-g} \equiv 1 \quad \text{(Implicitly Direct NHCES)}, \tag{2.5}$$

which is parameterized from Mukerji (1963)'s Constant Ratios of Elasticity of Substitution (CRES) model (with $g_i = g$ $\forall i$ in 2.6):

$$F(\mathbf{q}, u) = \sum_i k_i u^{-e_i(1-g_i)} q_i^{1-g_i} \equiv 1 \quad \text{(Implicitly Direct CRES)}, \tag{2.6}$$

where the parametric restrictions are

$$\begin{cases} k_i > 0 & \text{(i)} \\ e_i > 0 & \text{(ii)} \\ g_i > 0 & \text{(iii)} \\ g_i \geq 1 \quad \text{or} \quad 0 < g_i \leq 1 & \text{(iv)} \end{cases}$$

$\forall i$ for $u = f(\mathbf{q})$ in (2.6) to be globally valid (monotonic and quasi-concave).

[2]See pg.414 of Hanoch (1975).

For completeness, from the expenditure minimization problem to (2.6) (CRES): $\min\{\sum_i p_i q_i : \bar{F} \leq F(\mathbf{q}, u)\}$, the first-order conditions with respect to q_i give rise to:

$$p_i = \lambda k_i (1-g_i) u^{-e_i(1-g_i)} q_i^{-g_i} \qquad \forall i, \tag{2.7}$$

where

$$p_1 = \lambda k_1 (1-g_1) u^{-e_1(1-g_1)} q_1^{-g_1} \tag{2.8}$$

Dividing (2.7) by (2.8) eliminates $\lambda = \frac{\partial F(\mathbf{q},u)}{\partial q_i} \neq \frac{\partial f(\mathbf{q})}{\partial q_i} = \frac{\partial u}{\partial q_i}$, while yielding:

$$q_1^{-g_1} \frac{p_i}{p_1} = \frac{k_i(1-g_i)}{k_1(1-g_1)} u^{e_1(1-g_1)-e_i(1-g_i)} q_i^{-g_i}. \tag{2.9}$$

Solving for q_i:

$$q_i = (\frac{p_i}{p_1})^{-\frac{1}{g_i}} \left[\frac{k_i(1-g_i)}{k_1(1-g_1)}\right]^{\frac{1}{g_i}} u^{\frac{e_1(1-g_1)-e_i(1-g_i)}{g_i}} q_1^{\frac{g_1}{g_i}}. \tag{2.10}$$

Taking the natural logarithm of both sides of (2.10): [3]

(2.11)

$$\begin{aligned} \ln q_i &= \frac{1}{g_i} \ln\left[\frac{k_i(1-g_i)}{k_1(1-g_1)}\right] - \frac{1}{g_i}\ln(\frac{p_i}{p_1}) + \frac{e_1(1-g_1)-e_i(1-g_i)}{g_i} \ln u + \frac{g_1}{g_i} \ln q_1 \\ &= M_i - s_i \ln(\frac{p_i}{p_1}) + R_i \ln u + \frac{s_i}{s_1} \ln q_1 \quad \forall i \in [2, \infty), \end{aligned}$$

where $s_i = \frac{1}{g_i}$, $M_i = s_i \ln\left[\frac{k_i(1-g_i)}{k_1(1-g_1)}\right]$, and $R_i = s_i[e_1(1-g_1) - e_i(1-g_i)]$. Since the transformation of 2.6 (CRES) to 2.5 (Implicitly Direct NHCES) has arisen by restricting $g_i = g \ \forall i \implies s_i = s \ \forall i$, then (2.11) converges to:

$$\ln \frac{q_i}{q_1} = \widetilde{M}_i + \widetilde{R}_i \ln u - s \ln(\frac{p_i}{p_1}) \quad \forall i \in [2, \infty), \tag{2.12}$$

[3]The following proof is established using a linear approach (using log-differencing) based on Hanoch (1975). Yang (2019) demonstrates a direct non-linear approach (see Appendix D). It turns out that the resulting parametric requirements are slightly different from Hanoch. We show that, in addition to allow $\alpha = 1/g$, distribution parameter β_i in the CDE requires to be equate $k_i^\alpha \ \forall i$ (instead of $\beta_i = k_i \ \forall i$ in Hanoch), in order to transform implicitly indirect NHCES (as a special case generalized from CDE) to implicitly direct NHCES.

where $\widetilde{M}_i = s \ln \frac{k_i}{k_1}$, and $\widetilde{R}_i = (e_1 - e_i)(s-1)$;
which is identical to (2.4) $\iff$ $\widetilde{M}_i = \widetilde{A}_i$, $\widetilde{R}_i = \widetilde{Z}_i$, and $s = \alpha \implies \beta_i = k_i^{\alpha}\ \forall i$ and $\alpha = 1/g$.

Q.E.D.

To summarize,

$$\begin{cases} \text{CDE} \longrightarrow \text{Implicitly Indirect NHCES} & \iff \alpha_i = \alpha\, \forall i \\ \text{Implicitly Indirect NHCES} \equiv \text{Implicitly Direct NHCES} & \iff \alpha = 1/g \text{ and } \beta_i = k_i^{\alpha}\, \forall i \\ \text{Implicit CRES} \longrightarrow \text{Implicitly Direct NHCES} & \iff g_i = g\, \forall i \text{ and } \neg\, \forall e_i = e \end{cases}$$

3. EXPLICITLY INDIRECT HOMOTHETIC CES

As shown in Hanoch (1975), the specialized CDE can be parameterized to achieve a homothetic CES demand model. Such parameterization from CDE allows testings against the standard CES in GE trade models (e.g., Yang (2019)). It is common in GE to include an additional set of equilibrium conditions that characterizes a global economy, e.g., aggregate price index (see, e.g., Dixit and Stiglitz (1977) and the *real wealth assumption*. However, it can be challenging for some general demand systems to be parameterized in such a way that they always yield some desired equilibrium framework. We demonstrate a procedure under which circumstances that (1) CDE will converge to a homothetic CES demand system (as originally introduced by Hanoch); and (2) that additional parametric restrictions are required to be suitable in a GE framework, where there exists an aggregate price index and real wealth assumption is imposed.[4]

PROPOSITION 2 *Let $G(\boldsymbol{\xi}, u)$ be an implicitly indirectly additive utility function of Constant Difference of Elasticities (CDE), then $G(\boldsymbol{\xi}, u)$ can be parameterized to achieve an explicitly indirect constant Elasticity of Substitution (CES) function, which is identical to its explicitly direct case; it can be further parameterized to satisfy the standard CES price index, while satisfying the CES real wealth assumption in a general equilibrium framework.*

Definition 1: *A CES real wealth assumption is that price indices of aggregate goods consumed by a representative consumer, or cost of per capita utility, equates the per capita income adjusted by the per capita utility of the representative consumer, i.e.:*

$$P = w/u, \tag{3.1}$$

where P is the aggregate or consumer price index, w is the per capita income, and u is the per capita utility.

[4]The expression of an aggregate price index in CGE is virtually an assumption; also see Melitz (2003).

PROOF: Let $e_i = e$ and $\alpha_i = \alpha\ \forall i$, then (1.1) generalizes to:

$$G(\frac{\mathbf{p}}{w}, u) = \sum_i \beta_i u^{e(1-\alpha)} (\frac{p_i}{w})^{1-\alpha} \equiv 1. \tag{3.2}$$

(1) *Utility function is homogeneous* The income elasticity of generalized CDE function is:

$$\eta_i = \frac{e(1-\alpha) + \sum_k e\pi_k \alpha}{\sum_k e\pi_k} + \alpha - \sum_k \pi_k \alpha = 1. \tag{3.3}$$

(2) *Constant elasticity of substitution* The cross-Allen partial elasticity is:

$$\sigma_{ij} = \alpha_i + \alpha_j - \sum_k \pi_k \alpha_k - \frac{\Delta_{ij}\alpha_i}{\pi_i} = \alpha \quad \forall i \neq j. \tag{3.4}$$

(3) *Identical to Explicitly Direct CES* Rearranging (3.2) by factoring common terms:

$$u^{e(1-\alpha)} \sum_i \beta_i p_i^{1-\alpha} = w^{1-\alpha}, \tag{3.5}$$

which can be further simplified to:

$$\underbrace{u^e \Big(\sum_i \beta_i p_i^{1-\alpha} \Big)^{\frac{1}{1-\alpha}}}_{=P \iff e=1} = w, \tag{3.6}$$

and leads to the following explicitly indirect expression by isolating u:

$$U = \Big[\sum_i \beta_i \Big(\frac{p_i}{w} \Big)^{1-\alpha} \Big]^{\frac{1}{e(\alpha-1)}} \quad \text{(Explicitly Indirect Homothetic CES)}, \tag{3.7}$$

which is identical to the following explicitly direct homothetic CES:

$$U = \Big[\sum_{i=1}^{N} \beta_i^{\frac{1}{\alpha}} q_i^{\frac{\alpha-1}{\alpha}} \Big]^{\frac{\alpha}{e(\alpha-1)}} \quad \text{(Explicitly Direct Homothetic CES)}, \tag{3.8}$$

Suppose the identities take the form of (3.8), while global regularity conditions are satisfied as parametrically restricted in (1.1) except $\alpha \neq 1$. The utility maximization solving the problem: $\max\{u(\mathbf{q}) : \sum_i p_i q_i \leq w, i = 1, ..., n\}$, leads to the following demand function:

$$q_i = \frac{\beta_i p_i^{-\alpha}}{\sum_i \beta_i p_i^{1-\alpha}} w, \tag{3.9}$$

which is identical to (1.2) if $e_i = e$ and $\alpha_i = \alpha\ \forall i$. Also, substitution of (3.9) into first-order conditions of (3.8) yields the same price index as in the implicitly indirect case.[5]

(4) *CES Real wealth assumption can be further satisfied*

With $e = 1$ as a special version in (3.6), the utility function leads to the following:[6]

$$U = \Big[\sum_i \beta_i \Big(\frac{p_i}{w}\Big)^{1-\alpha}\Big]^{\frac{1}{\alpha-1}} \quad \text{(special case of 3.7)}. \tag{3.10}$$

P in (3.6) is the exact form of CES price indices defined in Dixit and Stiglitz (1977) based on Green (1964), with however the following direct homothetic CES preference:[7]

$$U = \Big[\sum_{i=1}^{N} \beta_i^{\frac{1}{\alpha}} q_i^{\frac{\alpha-1}{\alpha}}\Big]^{\frac{\alpha}{\alpha-1}} \quad \text{(special case of 3.8)}. \tag{3.11}$$

The ordinary demand from utility maximization to (3.11) yields the same result as in (3.9), which is derived from (3.7), invariant to whether $e = 1$ is additionally imposed. With the expression for the price index P in (3.6), the demand as a function of P in this special case can then be expressed as follows, which is equivalent as derived from (3.10):

$$q_i = \frac{\beta_i p_i^{-\alpha}}{P^{1-\alpha}} w, \tag{3.12}$$

Hence, a standard CDE as in (1.1) can be parameterized to achieve (3.7) of an explicitly indirect homothetic CES $\iff$ $e_i = e$, $\alpha_i = \alpha\ \forall\ i$, and $\alpha > 1$ or

[5]See Appendix C for complete mathematical derivations of Marshallian demand and price index for the explicitly direct CES. For the explicitly indirect CES, see section 4 for the price index derivation, and for the Marshallian demand it can be readily verified by parameterization to the derived CDE demand.

[6]See Appendix A for the derivation of Explicitly Indirect Homothetic CES.

[7]It turns out that the price index P solved from the explicit case is equivalent to the implicit case, which is computed from substituting U [derived from the total differentiation to (1.1)] to (3.8).

$0 < \alpha < 1$, which is identical to explicitly direct homothetic CES as in (3.8); and further satisfies (1) standard CES price index in GE; and (2) the real wealth assumption $\iff e = 1, \alpha_i = \alpha \; \forall \; i$. *Q.E.D.*

Parameterization to allow a standard CDE to transform into an indirect homothetic CES system automatically makes the demand model explicit. It is due to the fact that utility can be algebraically solved in terms of the model's exogenous variables and parameters, when α and e are both uniform.

4. PRICE INDICES

PROPOSITION 3 *Implicitly Indirect Non-Homothetic CDE, Implicit Homogeneous CDE and Implicitly Indirect Non-Homothetic CES systems all lead to aggregate price indices that are implicitly defined with endogenized utility; Explicitly Indirect Non-Homothetic CES leads to explicit price index, where utility index can be solved in terms of exogenous variables of the system.*

PROOF: *Non-Homothetic CDE* Suppose the utility function is implicitly defined as an implicitly indirect CDE in (1.1) and assume all global regularity conditions hold. Following Chen (2017), the total differentiation of (1.1) with respect to u and w at a given price vector, leads to the expression as follows:

(4.1)
$$\sum_i \beta_i e_i (1-\alpha_i) u^{e_i(1-\alpha_i)-1} (\frac{p_i}{w})^{1-\alpha_i} du + \sum_i (\alpha_i - 1) \beta_i u^{e_i(1-\alpha_i)} p_i^{1-\alpha_i} w^{\alpha_i - 2} dw \equiv 0.$$

Then the marginal cost of utility can be derived as:

(4.2)
$$\frac{dw}{du} = \left[\sum_i \beta_i u^{e_i - e_i\alpha_i - 1} (1-\alpha_i) p_i^{1-\alpha_i} w^{\alpha_i - 1} e_i \right] \left[\sum_i \beta_i u^{e_i(1-\alpha_i)} (1-\alpha_i) p_i^{1-\alpha_i} w^{\alpha_i - 2} \right]^{-1},$$

where $\frac{dw}{du} = P \equiv \lambda^{-1}$, P is the aggregate price index; and λ is the Lagrange multiplier (from the utility maximization problem as if it were solved from the explicit case), representing marginal utility of income, as can be solved from the utility maximization to any explicit direct utility functions: $\max\{u(\mathbf{q}) : p'q \leq w\}$ with its gradient vector evaluated at $\mathbf{q}$ at an interior optimum.[8]

[8]The consumer price index $P \equiv \lambda^{-1}$ can be further verified in Dixit and Stiglitz (1977) and Green (1964). In Appendix B, we show a simple example of the version of Dixit-Stiglitz two goods; Appendix C is indeed a more generalized case with N goods with the expansion parameter $e \neq 1$.

Given (1.1), (4.2) can be rewritten in a reduced form using the following expression:

$$P = \eta \frac{w}{u}, \tag{4.3}$$

where $\eta = \sum_i \pi_i e_i$ is the elasticity of aggregate expenditure with respect to utility, and π_i is the optimal expenditure share. Since the expenditure share is a function of an ordinal demand in (1.2) where its utility is endogenized, and the irreducible summation cannot be factored out over e_i, then P is implicitly defined, depending on the utility.

Implicitly Indirect NHCES By proposition 2, an implicitly indirect CDE will converge to an implicitly indirect CES if $\alpha_i = \alpha \ \forall i$. Using (4.2), then the price index is expressed as:

$$P = \frac{\sum_i \beta_i u^{e_i(1-\alpha)} p_i^{1-\alpha} e_i}{\sum_i \beta_i u^{e_i(1-\alpha)} p_i^{1-\alpha}} \frac{w}{u}, \tag{4.4}$$

which is implicitly defined with the utility u, as in the case of non-homothetic CDE.

Implicit Homogeneous CDE Again, by Proposition 2, an implicitly indirect CDE can be parameterized to transform to an implicit homogeneous CDE by allowing $e_i = e \ \forall i$, then the price index can be expressed as:

$$P = e \frac{w}{u}, \tag{4.5}$$

which appears to be a succinct functional form, but cannot be further simplified with elimination of u, which is endogenously determined even $e_i = e \ \forall i$.

Explicitly Indirect Homothetic CES We know that the price index takes a reduced form in (4.5) if $e_i = e \ \forall i$, then in the case of explicitly indirect non-homothetic CES, the price index yields the same expression when, additionally, allowing $\alpha_i = \alpha \ \forall i$, as can be also verified in (4.4) if e is factored out of the summation.

From the identity in (3.5), it is readily demonstrable that the price index can be algebraically solved in this explicit case. Rearranging (4.5) and then substituting $u = \frac{ew}{P}$ into (3.6) immediately yields the following expression:

$$\left(\frac{ew}{P}\right)^e \left(\sum_i \beta_i p_i^{1-\alpha}\right)^{\frac{1}{1-\alpha}} = w. \tag{4.6}$$

Isolating P to derive the price index of the exact form:

$$
\begin{aligned}
P &= e\big[\sum_i \beta_i p_i^{1-\alpha} w^{(e-1)(1-\alpha)}\big]^{\frac{1}{e(1-\alpha)}} \\
&= e\big[\sum_i \beta_i (\frac{p_i}{w^{1-e}})^{1-\alpha}\big]^{\frac{1}{e(1-\alpha)}},
\end{aligned} \tag{4.7}
$$

Q.E.D.

It is easy to see from (4.7), the price index leads to the same result (shown under the curly brackets in 3.6) as derived from the explicitly direct case in (3.8) (see proof in appendix C).

5. EXPENDITURE SHARES

Implicitly Indirect Non-Homothetic CDE With $\pi_i = \frac{p_i q_i}{w}$ and ordinary demand in (1.2), the per capita expenditure share of i in the standard CDE can be expressed as the following:

$$\pi_i = \frac{\beta_i(1-\alpha_i)u^{e_i(1-\alpha_i)}(\frac{p_i}{w})^{1-\alpha_i}}{\sum_j \beta_j(1-\alpha_j)u^{e_j(1-\alpha_j)}(\frac{p_j}{w})^{1-\alpha_j}}. \tag{5.1}$$

Implicitly Indirect Homogeneous CDE With $e_i = e \ \forall i$, the standard CDE is homogenous. The expenditure share leads to the following expression:

$$\pi_i = \frac{\beta_i(1-\alpha_i)u^{-e\alpha_i}(\frac{p_i}{w})^{1-\alpha_i}}{\sum_j \beta_j(1-\alpha_j)u^{-e\alpha_j}(\frac{p_j}{w})^{1-\alpha_j}}. \tag{5.2}$$

Implicitly Indirect NHCES By proposition 1, the CDE function converges to Implicitly Indirect NHCES $\iff$ $\alpha_i = \alpha \ \forall i$. Its expenditure share leads to the following expression:

$$\pi_i = \frac{\beta_i u^{e_i(1-\alpha)} p_i^{1-\alpha}}{\sum_j \beta_j u^{e_j(1-\alpha)} p_j^{1-\alpha}}. \tag{5.3}$$

(5.3) shows that expenditure shares in the implicit NHCES are not directly affected by changes in income, although they are responsive to incomes where utility is endogenized and is implicitly defined as a function of incomes.

Explicitly Indirect Homothetic CES
By proposition 2, convergence to homothetic CES requires that $e_i = e \; \forall i$, which leads to the following expenditure shares expression:[9]

$$\pi_i = \frac{\beta_i p_i^{1-\alpha}}{\sum_j \beta_j p_j^{1-\alpha}} \implies \beta_i \left(\frac{P}{p_i}\right)^{\alpha-1} \iff e = 1. \tag{5.4}$$

(5.3) shows that the expenditure shares under implicit NHCES are only affected by changes in the price vector, but are independent of any income changes.

6. COUNTERFACTUAL WELFARE RESPONSES

Implicit Indirect Non-Homothetic CDE
Implementing total differentiation in (1.1) with respect to utility, wealth and the price vector:

$$\begin{aligned} &\sum_i \beta_i e_i (1-\alpha_i) u^{e_i(1-\alpha_i)-1} (\frac{p_i}{w})^{1-\alpha_i} du \\ &+ \sum_i (\alpha_i - 1)\beta_i u^{e_i(1-\alpha_i)} p_i^{1-\alpha_i} w^{\alpha_i - 2} dw \\ &+ \sum_i \beta_i (1-\alpha_i) u^{e_i(1-\alpha_i)} p_i^{-\alpha_i} w^{\alpha_i - 1} dp_i \\ &\equiv \quad 0. \end{aligned} \tag{6.1}$$

Equation (6.1) can be simplified as:

$$\begin{aligned} &\sum_i (1-\alpha_i)\beta_i u^{e_i(1-\alpha_i)} (\frac{p_i}{w})^{1-\alpha_i} \overbrace{\frac{dw}{w}}^{\text{Change of Wealth}} \\ \equiv &\sum_i \beta_i e_i (1-\alpha_i) u^{e_i(1-\alpha_i)} (\frac{p_i}{w})^{1-\alpha_i} \underbrace{\overbrace{\frac{du}{u}}}_{\text{Change of Utility}} \\ &+ \sum_i \beta_i (1-\alpha_i) u^{e_i(1-\alpha_i)} (\frac{p_i}{w})^{1-\alpha_i} \underbrace{\overbrace{\frac{dp_i}{p_i}}}_{\text{Change of Price}} \end{aligned} \tag{6.2}$$

Rewriting the CDE expenditure share expression in (5.1), e.g., $\frac{p_i q_i}{\sum_i p_i q_i}$:

$$\pi_i = \frac{\beta_i u^{e_i(1-\alpha_i)} (1-\alpha_i) (\frac{p_i}{w})^{1-\alpha_i}}{\sum_j \beta_j u^{e_j(1-\alpha_j)} (1-\alpha_j) (\frac{p_j}{w})^{1-\alpha_j}} = \frac{\beta_i u^{e_i(1-\alpha_i)} (1-\alpha_i) (\frac{p_i}{w})^{1-\alpha_i}}{T}. \tag{6.3}$$

[9] P is the price index with respect to (3.7).

Note that if we divide both sides by T of equation (6.2), and using *hat* to denote rate of changes on corresponding terms, then the expression can be further simplified to:

$$\sum_i \pi_i \hat{w} \equiv \sum_i e_i \pi_i \hat{u} + \sum_i \pi_i \hat{p}_i, \tag{6.4}$$

and since $\sum_i \pi_i = 1$, and $\hat{w}$ does not depend on each i, we obtain:

$$\hat{w} \equiv \sum_i e_i \pi_i \hat{u} + \sum_i \pi_i \hat{p}_i, \tag{6.5}$$

and because $\hat{u}$ does not depend on each i, the change of utility can be written as:

$$\hat{u} = \frac{\hat{w} - \sum_i \pi_i \hat{p}_i}{\sum_i e_i \pi_i}. \tag{6.6}$$

where $\pi_i = \dfrac{\beta_i(1-\alpha_i) \overbrace{u^{e_i(1-\alpha_i)}}^{\text{Effects of Baseline Utility}} \overbrace{(\frac{p_i}{w})^{1-\alpha_i}}^{\text{Effects of Baseline Normalized Prices}}}{\underbrace{\sum_j \beta_j(1-\alpha_j) u^{e_j(1-\alpha_j)} (\frac{p_j}{w})^{1-\alpha_j}}_{\text{Composite Effects of Baseline Utility, Income and Prices}}}$ following (5.1).

In this case, change of cardinal utility does not only respond to changes of income and price vectors, but is also determined by heterogeneous commodity-specific CDE parameters as well as baseline income, utility and commodity prices.

Implicitly Indirect Homogeneous CDE In the case of homogenous CDE where $e_i = e \ \forall i$, the denominator in (6.6) converges to the uniform expansion parameter e, where $\pi_i = \frac{\beta_i(1-\alpha_i)u^{-e\alpha_i}(\frac{p_i}{w})^{1-\alpha_i}}{\sum_j \beta_j(1-\alpha_j)u^{-e\alpha_j}(\frac{p_j}{w})^{1-\alpha_j}}$ that follows (5.2):

$$\hat{u} = \frac{\hat{w} - \sum_i \pi_i \hat{p}_i}{e} \tag{6.7}$$

Implicitly Indirect NHCES With uniform α, the expression in (6.6) cannot be further simplified, with however the expenditure share $\pi_i = \frac{\beta_i u^{e_i(1-\alpha)} p_i^{1-\alpha}}{\sum_j \beta_j u^{e_j(1-\alpha)} p_j^{1-\alpha}}$ that follows (5.3). Comparing with the standard CDE, the implicitly indirect NHCES eliminates the effects of baseline per capita income.

Explicitly Indirect Homothetic CES The only possible way of parameterization to make the CDE utility explicitly derivable (otherwise remains to be implicitly defined) leads to an explicitly indirect Constant elasticity of substitution (CES), if allowing $e_i = e$ and $\alpha_i = \alpha\ \forall i$. The same formula for change of utility (6.7) can be applied, with expenditure share $\pi_i = \frac{\beta_i p_i^{1-\alpha}}{\sum_j \beta_j p_j^{1-\alpha}}$ that follows (5.4). In this case, the change of utility is invariant to effects of both baseline utility and income levels.

It turns out that, by transforming an implicitly indirect CDE to an explicitly indirect homothetic CES, the consumer welfare appears to be *more responsive* to counterfactual income changes, lying in the fact that both effects of baseline utility and income levels are *eliminated*. None of the cases above, however, eliminates the effects of baseline prices on changes of utility, even with the special case of CES where $e = 1$.

7. COUNTERFACTUAL TRADE RESPONSES

Implicit Indirect Non-Homothetic CDE Similarly, taking total derivatives with respect to implicit utility, price vectors, and per capita income will lead to the following percent change expression:

(7.1)
$$\begin{aligned}
\hat{q}_i &= e_i(1-\alpha_i)\hat{u} - \alpha_i\hat{p}_i + \alpha_i\hat{w} - \sum_j e_j(1-\alpha_j)\pi_j\hat{u} - \sum_j(1-\alpha_j)\pi_j\hat{p}_j + \sum_j(1-\alpha_j)\pi_j\hat{w} \\
&= \left[e_i(1-\alpha_i) - \sum_j e_j(1-\alpha_j)\pi_j\right]\hat{u} \\
&+ \left[\alpha_i + \sum_j(1-\alpha_j)\pi_j\right]\hat{w} \\
&- \alpha_i\hat{p}_i - \sum_j(1-\alpha_j)\pi_j\hat{p}_j.
\end{aligned}$$

The analysis of quantity consumption is non-trivial and (7.1) can be considered as a special case where there are zero trade costs and f.o.b. prices are normalized to one (if we care more about the counterfactual price changes but not prices at their initial values). The counterfactual result of quantity consumption depends on utility and income changes, as well as price changes of own-goods and all other goods bundle. Meanwhile changes in utility, income and prices are interacted with expansion and substitution parameters, e_i and α_i, respectively with respect to goods i, as well as with share-weighted parameter values of the two with respect to all commodities $\forall i \in I$.

Substituting (6.2) into (7.1), the change of quantity consumption can also be expressed as a function of income and cross-price elasticities:

$$\hat{q}_i = \eta_i \hat{w} + \sum_j \sigma_{i,j} \hat{p}_j, \tag{7.2}$$

where $\eta_i = \frac{e(1-\alpha)+\sum_k e\pi_k\alpha}{\sum_k e\pi_k} + \alpha - \sum_k \pi_k \alpha$ and $\sigma_{i,j} = \alpha_i + \alpha_j - \sum_k \pi_k \alpha_k - \frac{\Delta_{ij}\alpha_i}{\pi_i}$, which follows (3.3) and (3.4), respectively.

Explicitly Indirect Homothetic CES Generalized explicit case removes the *effects of utility change* in (7.1):

$$\hat{q}_i = \hat{w} - \alpha \hat{p}_i - (1-\alpha) \sum_j \hat{p}_j, \tag{7.3}$$

which is identical to the counterfactual result of the standard CES.

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APPENDIX A: DERIVATION OF EXPLICITLY INDIRECT HOMOTHETIC CES

With $e_i = e$ and $\alpha_i = \alpha \; \forall i$, equation (1.1) automatically yields an explicit demand function and leads to the following expression from the G function of the CDE:

$$U = \left[\frac{w}{\left(\sum_i \beta_i p_i^{1-\alpha}\right)^{\frac{1}{1-\alpha}}} \right]^{\frac{1}{e}}. \tag{A.1}$$

Rearranging terms leads to explicitly indirect NHCES (derived from the CDE):

$$
\begin{aligned}
U &= \left[\frac{w^{1-\alpha}}{\sum_i \beta_i p_i^{1-\alpha}}\right]^{\frac{1}{e(1-\alpha)}} \\
&= \left[\frac{w^{\alpha-1}}{(\sum_i \beta_i p_i^{1-\alpha})^{-1}}\right]^{\frac{1}{e(\alpha-1)}} \\
&= \left[w^{\alpha-1} \sum_i \beta_i p_i^{1-\alpha}\right]^{\frac{1}{e(\alpha-1)}} \\
&= \left[\sum_i \beta_i \left(\frac{p_i}{w}\right)^{1-\alpha}\right]^{\frac{1}{e(\alpha-1)}}.
\end{aligned} \tag{A.2}
$$

With $e = 1$, the expression further converges to:

$$U = \left[\sum_i \beta_i \left(\frac{p_i}{w}\right)^{1-\alpha}\right]^{\frac{1}{\alpha-1}}. \tag{A.3}$$

APPENDIX B: DIXIT-STIGLITZ TWO GOODS (EXPLICITLY DIRECT CES)

The Lagrangian function to a version of two-goods of Dixit and Stiglitz (1977) is given by:

$$L = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}} - \lambda(x_1 p_1 + x_2 p_2 - y) \tag{B.1}$$

F.O.C. with respect to x_1, x_2 and λ yields (B.2)-(B.4) as follows:

$$\frac{\partial L}{\partial x_1} = (x_1^\rho + x_2^\rho)^{\frac{1-\rho}{\rho}} x_1^{\rho-1} = \lambda p_1 \tag{B.2}$$

$$\frac{\partial L}{\partial x_2} = (x_1^\rho + x_2^\rho)^{\frac{1-\rho}{\rho}} x_2^{\rho-1} = \lambda p_2 \tag{B.3}$$

$$y = x_1 p_1 + x_2 p_2 \tag{B.4}$$

Dividing (B.2) by (B.3) leads to (B.5)-(B.8):

$$\frac{x_1^{\rho-1}}{x_2^{\rho-1}} = \frac{p_1}{p_2} \tag{B.5}$$

$$\frac{x_1}{x_2} = (\frac{p_1}{p_2})^{\frac{1}{\rho-1}} \tag{B.6}$$

$$x_1 = (\frac{p_1}{p_2})^{\frac{1}{\rho-1}} x_2 \tag{B.7}$$

$$x_2 = \frac{x_1}{(\frac{p_1}{p_2})^{\frac{1}{\rho-1}}} = (\frac{p_1}{p_2})^{\frac{1}{1-\rho}} x_1 \tag{B.8}$$

plugging (B.7) to the budget constraint, or (B.4) gives the following expression:

$$\begin{aligned}
y &= (\frac{p_1}{p_2})^{\frac{1}{\rho-1}} x_2^* p_1 + x_2^* p_2 \\
&= x_2^*[(\frac{p_1}{p_2})^{\frac{1}{\rho-1}} p_1 + p_2] \\
&= x_2^*(p_1^{\frac{1}{\rho-1}} p_1 p_2^{-\frac{1}{\rho-1}} + p_2) \\
&= x_2^*(p_1^{\frac{\rho}{\rho-1}} p_2^{\frac{1}{\rho-1}} + p_2) \\
&= x_2^* p_2^{\frac{1}{1-\rho}} (p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}})
\end{aligned} \tag{B.9}$$

Solving for the optimal demand for x_2:

$$x_2^* = \frac{y p_2^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}} \tag{B.10}$$

plugging (B.8) to to the budget constraint, or (B.4):

$$\begin{aligned}
y &= x_1^* p_1 + (\frac{p_1}{p_2})^{\frac{1}{1-\rho}} x_1^* p_2 \\
&= x_1^*[p_1 + (\frac{p_1}{p_2})^{\frac{1}{1-\rho}} p_2] \\
&= x_1^*(p_1^{\frac{1}{1-\rho}} p_2^{\frac{1}{\rho-1}} p_2 + p_1) \\
&= x_1^*(p_1^{\frac{1}{1-\rho}} p_2^{\frac{\rho}{\rho-1}} + p_1) \\
&= x_2^* p_1^{\frac{1}{1-\rho}} (p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}})
\end{aligned} \tag{B.11}$$

Solving for x_1^*:

$$x_1^* = \frac{y p_1^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}} \tag{B.12}$$

Plugging (B.10) and (B.12) to (B.2), yielding equivalent results as plugging to (B.3):

$$\begin{aligned}
\lambda p_1 &= \Big[\Big(\frac{y p_1^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}}\Big)^{\rho} + \Big(\frac{y p_2^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}}\Big)^{\rho}\Big]^{\frac{1-\rho}{\rho}} \Big[\frac{y p_1^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}}\Big]^{\rho-1} \\
&= \Big[\frac{y^{\rho} p_1^{\frac{\rho}{\rho-1}} + y^{\rho} p_2^{\frac{\rho}{\rho-1}}}{(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}})^{\rho}}\Big]^{\frac{1-\rho}{\rho}} \Big[\frac{y^{\rho-1} p_1}{(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}})^{\rho-1}}\Big] \\
&= \Big[\frac{y^{1-\rho} (p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}})^{\frac{1-\rho}{\rho}}}{(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}})^{1-\rho}}\Big] \Big[\frac{y^{\rho-1} p_1}{(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}})^{\rho-1}}\Big] \\
&= (p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}})^{\frac{1-\rho}{\rho}} p_1
\end{aligned} \tag{B.13}$$

Solving for λ:

$$\lambda = (p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}})^{\frac{1-\rho}{\rho}} \tag{B.14}$$

Solving for λ^{-1}:

$$\begin{aligned}
\lambda^{-1} &= \Big[(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}})^{\frac{1-\rho}{\rho}}\Big]^{-1} \\
&= (p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}})^{\frac{\rho-1}{\rho}} \\
&= (p_1^{\frac{-1}{\beta}} + p_2^{\frac{-1}{\beta}})^{-\beta}
\end{aligned} \tag{B.15}$$

where $\beta = (1-\rho)/\rho$, and $0 < \rho < 1$, so $\beta > 0$,

which shows that the aggregate price index defined in Dixit and Stiglitz (1977) based on Green (1964) is the *marginal cost of utility* (which is the price of utility, or referred to as aggregate price index in some other literature, e.g., λ^{-1}).

APPENDIX C: EXPLICITLY DIRECT CES

Suppose the utility function takes the following form:

$$U = \Big[\sum_{i=1}^{N} \beta_i^{\frac{1}{\alpha}} q_i^{\frac{\alpha-1}{\alpha}}\Big]^{\frac{\alpha}{e(\alpha-1)}} \tag{C.1}$$

Maximizing U subject to consumer budget constraint: $\sum_i p_i q_i \leq w$, then the Lagrangian function is written as:

$$L = \Big[\sum_{i=1}^{N} \beta_i^{\frac{1}{\alpha}} q_i^{\frac{\alpha-1}{\alpha}}\Big]^{\frac{\alpha}{e(\alpha-1)}} - \lambda(\sum_i p_i q_i - w). \tag{C.2}$$

F.O.C. with respect to q_i gives the following result:

$$\begin{aligned}\frac{\partial L}{\partial q_i} &= \frac{\alpha}{e(\alpha-1)}\Big[\sum_{i=1}^{N}\beta_i^{\frac{1}{\alpha}}q_i^{\frac{\alpha-1}{\alpha}}\Big]^{\frac{\alpha-e(\alpha-1)}{e(\alpha-1)}}\beta_i^{\frac{1}{\alpha}}\frac{\alpha-1}{\alpha}q_i^{-\frac{1}{\alpha}}-\lambda p_i \\ &= \frac{1}{e}\Big[\sum_{i=1}^{N}\beta_i^{\frac{1}{\alpha}}q_i^{\frac{\alpha-1}{\alpha}}\Big]^{\frac{\alpha-e(\alpha-1)}{e(\alpha-1)}}\beta_i^{\frac{1}{\alpha}}q_i^{-\frac{1}{\alpha}}-\lambda p_i \\ &= 0,\end{aligned} \tag{C.3}$$

with λp_i that equates:

$$\lambda p_i = \frac{1}{e}\Big[\sum_{i=1}^{N}\beta_i^{\frac{1}{\alpha}}q_i^{\frac{\alpha-1}{\alpha}}\Big]^{\frac{\alpha-e(\alpha-1)}{e(\alpha-1)}}\beta_i^{\frac{1}{\alpha}}q_i^{-\frac{1}{\alpha}}, \tag{C.4}$$

which also implies that:

$$\lambda p_1 = \frac{1}{e}\Big[\sum_{i=1}^{N}\beta_i^{\frac{1}{\alpha}}q_i^{\frac{\alpha-1}{\alpha}}\Big]^{\frac{\alpha-e(\alpha-1)}{e(\alpha-1)}}\beta_1^{\frac{1}{\alpha}}q_1^{-\frac{1}{\alpha}}, \tag{C.5}$$

Dividing both sides of the two F.O.C., yields:

$$\frac{p_i}{p_1} = (\frac{\beta_i}{\beta_1})^{\frac{1}{\alpha}}(\frac{q_i}{q_1})^{-\frac{1}{\alpha}}. \tag{C.6}$$

Solving for q_i:

$$q_i = \frac{\beta_i}{\beta_1}(\frac{p_i}{p_1})^{-\alpha}q_1. \tag{C.7}$$

Then the expenditure function of q_i can be expressed as:

$$p_i q_i = \frac{\beta_i}{\beta_1}p_i^{1-\alpha}p_1^{\alpha}q_1, \tag{C.8}$$

which implies that:

$$w = \sum_i p_i q_i = \beta_1^{-1}\sum_i \beta_i p_i^{1-\alpha}p_1^{\alpha}q_1. \tag{C.9}$$

Solving for q_1:

$$q_1 = \frac{w\beta_1 p_1^{-\alpha}}{\sum_i \beta_i p_i^{1-\alpha}}. \tag{C.10}$$

Substituting for q_i:

$$q_i = \frac{\beta_i p_i^{-\alpha}}{\sum_i \beta_i p_i^{1-\alpha}} w. \tag{C.11}$$

Explicitly Direct CES Price Index Substituting q_i into (C.4), with:

$$\begin{aligned}
\lambda p_i &= \frac{1}{e}\Big[\sum_{i=1}^{N} \beta_i^{\frac{1}{\alpha}} \big(\frac{\beta_i p_i^{-\alpha}}{\sum_i \beta_i p_i^{1-\alpha}} w\big)^{\frac{\alpha-1}{\alpha}}\Big]^{\frac{\alpha-e(\alpha-1)}{e(\alpha-1)}} \beta_i^{\frac{1}{\alpha}} \big(\frac{\beta_i p_i^{-\alpha}}{\sum_i \beta_i p_i^{1-\alpha}} w\big)^{-\frac{1}{\alpha}} \\
&= \frac{1}{e}\Big[\sum_{i=1}^{N} \beta_i^{\frac{1}{\alpha}} \big(\frac{\beta_i p_i^{-\alpha}}{\sum_i \beta_i p_i^{1-\alpha}}\big)^{\frac{\alpha-1}{\alpha}}\Big]^{\frac{\alpha-e(\alpha-1)}{e(\alpha-1)}} \frac{p_i}{\big(\sum_i \beta_i p_i^{1-\alpha}\big)^{-\frac{1}{\alpha}}} w^{\frac{1-e}{e}} \\
&= \frac{1}{e}\Big[\sum_{i=1}^{N} \frac{\beta_i p_i^{1-\alpha}}{\big(\sum_i \beta_i p_i^{1-\alpha}\big)^{\frac{\alpha-1}{\alpha}}}\Big]^{\frac{\alpha-e(\alpha-1)}{e(\alpha-1)}} \frac{p_i}{\big(\sum_i \beta_i p_i^{1-\alpha}\big)^{-\frac{1}{\alpha}}} w^{\frac{1-e}{e}} \\
&= \frac{1}{e} \frac{\big(\sum_i \beta_i p_i^{1-\alpha}\big)^{\frac{\alpha-e(\alpha-1)}{e(\alpha-1)}}}{\big(\sum_i \beta_i p_i^{1-\alpha}\big)^{\frac{\alpha-e(\alpha-1)}{\alpha e}}} \Big[\sum_i \beta_i p_i^{1-\alpha}\Big]^{\frac{1}{\alpha}} \frac{1}{w^{\frac{e-1}{e}}} p_i \\
&= \frac{1}{e}\Big[\sum_i \beta_i p_i^{1-\alpha}\Big]^{\frac{1}{e(\alpha-1)}} \frac{1}{w^{\frac{e-1}{e}}} p_i
\end{aligned} \tag{C.12}$$

Solving for $P = \lambda^{-1}$:

$$\begin{aligned}
P &= e\Big[\sum_i \beta_i p_i^{1-\alpha}\Big]^{\frac{1}{e(1-\alpha)}} w^{\frac{e-1}{e}} \\
&= e\Big[\sum_i \beta_i p_i^{1-\alpha} w^{(e-1)(1-\alpha)}\Big]^{\frac{1}{e(\alpha-1)}} \\
&= e\big[\sum_i \beta_i \big(\frac{p_i}{w^{1-e}}\big)^{1-\alpha}\big]^{\frac{1}{e(1-\alpha)}}.
\end{aligned} \tag{C.13}$$

which is identical to the price index solved from the case of implicitly indirect CES.

APPENDIX D: IMPLICITLY DIRECT NHCES

Implicitly Direct CRES From the expenditure minimization problem to (2.6) (Implicitly Direct CRES): $\min\{\sum_i p_i q_i : \bar{F} \le F(\mathbf{q}, u)\}$, the first-order conditions with respect to q_i give rise to (see also Hanoch (1971)):

$$p_i = \lambda k_i (1-g_i) u^{-e_i(1-g_i)} q_i^{-g_i} \qquad \forall i, \tag{D.1}$$

where

$$p_1 = \lambda k_1(1-g_1)u^{-e_1(1-g_1)}q_1^{-g_1} \tag{D.2}$$

Dividing the two equations above eliminates $\lambda = \frac{\partial F(\mathbf{q},u)}{\partial q_i} \neq \frac{\partial f(\mathbf{q})}{\partial q_i} = \frac{\partial u}{\partial q_i}$, while yielding:

$$q_1^{-g_1}\frac{p_i}{p_1} = \frac{k_i(1-g_i)}{k_1(1-g_1)}u^{e_1(1-g_1)-e_i(1-g_i)}q_i^{-g_i}. \tag{D.3}$$

Solving for q_i:

$$q_i = (\frac{p_i}{p_1})^{-\frac{1}{g_i}}\left[\frac{k_i(1-g_i)}{k_1(1-g_1)}\right]^{\frac{1}{g_i}} u^{\frac{e_1(1-g_1)-e_i(1-g_i)}{g_i}} q_1^{\frac{g_1}{g_i}}, \tag{D.4}$$

which implies that:

$$p_iq_i = p_i^{\frac{g_i-1}{g_i}} p_1^{\frac{1}{g_i}}\left[\frac{k_i(1-g_i)}{k_1(1-g_1)}\right]^{\frac{1}{g_i}} u^{\frac{e_1(1-g_1)-e_i(1-g_i)}{g_i}} q_1^{\frac{g_1}{g_i}}. \tag{D.5}$$

The total expenditure can be expressed as:

$$\sum_i p_iq_i = w = \sum_i p_i^{\frac{g_i-1}{g_i}} p_1^{\frac{1}{g_i}}\left[\frac{k_i(1-g_i)}{k_1(1-g_1)}\right]^{\frac{1}{g_i}} u^{\frac{e_1(1-g_1)-e_i(1-g_i)}{g_i}} q_1^{\frac{g_1}{g_i}}. \tag{D.6}$$

Solving for q_1:

$$q_1 = \frac{w^{\frac{g_i}{g_1}}}{\left[\sum_i p_i^{\frac{g_i-1}{g_i}} p_1^{\frac{1}{g_i}}\big(\frac{k_i(1-g_i)}{k_1(1-g_1)}\big)^{\frac{1}{g_i}} u^{\frac{e_1(1-g_1)-e_i(1-g_i)}{g_i}}\right]^{\frac{g_i}{g_1}}} \tag{D.7}$$

Substituting back to the first-order expression for q_i, while eliminating q_1:

(D.8)

$$q_i = (\frac{p_i}{p_1})^{-\frac{1}{g_i}}\left[\frac{k_i(1-g_i)}{k_1(1-g_1)}\right]^{\frac{1}{g_i}} u^{\frac{e_1(1-g_1)-e_i(1-g_i)}{g_i}} \frac{w}{\sum_i p_i^{\frac{g_i-1}{g_i}} p_1^{\frac{1}{g_i}}\big(\frac{k_i(1-g_i)}{k_1(1-g_1)}\big)^{\frac{1}{g_i}} u^{\frac{e_1(1-g_1)-e_i(1-g_i)}{g_i}}}$$

$$= \frac{(\frac{p_i}{p_1})^{-\frac{1}{g_i}}\big[\frac{k_i(1-g_i)}{k_1(1-g_1)}\big]^{\frac{1}{g_i}} u^{\frac{e_1(1-g_1)-e_i(1-g_i)}{g_i}} w}{\sum_i p_i^{\frac{g_i-1}{g_i}} p_1^{\frac{1}{g_i}}\big(\frac{k_i(1-g_i)}{k_1(1-g_1)}\big)^{\frac{1}{g_i}} u^{\frac{e_1(1-g_1)-e_i(1-g_i)}{g_i}}}.$$

Implicitly Direct NHCES If $g_i = g\ \forall i$, then the implicitly Direct CRES generalizes to implicitly direct NHCES. Then we may directly apply this parameterization to (D.8) and solves for q_i:

$$
\begin{aligned}
q_i &= \frac{\left(\frac{p_i}{p_1}\right)^{-\frac{1}{g_i}}\left[\frac{k_i(1-g_i)}{k_1(1-g_1)}\right]^{\frac{1}{g_i}} u^{\frac{e_1(1-g_1)-e_i(1-g_i)}{g_i}} w}{\sum_i p_i^{\frac{g_i-1}{g_i}} p_1^{\frac{1}{g_i}} \left(\frac{k_i(1-g_i)}{k_1(1-g_1)}\right)^{\frac{1}{g_i}} u^{\frac{e_1(1-g_1)-e_i(1-g_i)}{g_i}}} \\
&= \frac{p_i^{-\frac{1}{g}}\left(\frac{k_i}{k_1}\right)^{\frac{1}{g}} u^{\frac{(e_1-e_i)(1-g)}{g}} w}{\sum_i p_i^{\frac{g-1}{g}}\left(\frac{k_i}{k_1}\right)^{\frac{1}{g}} u^{\frac{(e_1-e_i)(1-g)}{g}}} \\
&= \frac{p_i^{-\frac{1}{g}}\left(\frac{k_i}{k_1}\right)^{\frac{1}{g}} u^{\frac{(e_1-e_i)(1-g)}{g}} w}{\sum_i p_i^{\frac{g-1}{g}}\left(\frac{k_i}{k_1}\right)^{\frac{1}{g}} u^{\frac{(e_1-e_i)(1-g)}{g}}} \\
&= \frac{k_i^{\frac{1}{g}} u^{\frac{e_i(g-1)}{g}} p_i^{-\frac{1}{g}} w}{\sum_i k_i^{\frac{1}{g}} u^{\frac{e_i(g-1)}{g}} p_i^{\frac{g-1}{g}}}
\end{aligned} \tag{D.9}
$$

Let $g = 1/\alpha$, then we may rewrite (D.9) as follows:

$$
q_i = \frac{k_i^{\alpha} u^{e_i(1-\alpha)} p_i^{-\alpha} w}{\sum_i k_i^{\alpha} u^{e_i(1-\alpha)} p_i^{1-\alpha}} \tag{D.10}
$$

If we additionally allow $k_i = \beta_i^{1/\alpha} = \beta_i^{g}$, then implicitly direct NHCES is identical to implicitly indirect NHCES, which is generalized by letting $\alpha_i = \alpha \ \forall i$ in the standard CDE, and leads to q_i to the following expression:

$$
q_i = \frac{\beta_i u^{e_i(1-\alpha)} p_i^{-\alpha} w}{\sum_i \beta_i u^{e_i(1-\alpha)} p_i^{1-\alpha}}, \tag{D.11}
$$

which is the same result using linear approach in section 2.