Theory and Methods for Computing General Equilibrium (GE) Models

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February 15, 2023

Yale Macroeconomics Seminar

Summary

- Different (non-text-book) angle to look at Computable General Equilibrium (CGE) models (JMP and work with Costas).
- Different view on bridging the CGE models with other trade models, and IO models (some work with Steve and Costas).
- New method to the (counterfactual) solution of these models.

Preliminaries

- We ask "what-if" questions in GE trade literature (economics).
- Ate too much breakfast? Feel too full? Gain weight?
- There are many similar tools to do this, e.g., Markusen (1984), Rutherford (1995), Balistreri and Hillberry (2007), Deckle, Eaton and Kortum (2008), Arkolakis, Costinot and Rodríguez-Clare (2012), and the GTAP/other CGE models largely expanded since 1990s'.
- If the model is getting larger and more complex, e.g., implicit models, then some of the tools may be hard to solve.
- We review and compare different approaches, e.g., exact hat algebra and MCP methods of counterfactual computation.
- We discuss an alternative computation method.
- We bridge exact hat algebra with MCP approach, then with the new method.

What If A Kitty Eats Too Much

Source: Potty Pot Pie

Counterfactual Consequence

- Input statement $\Longrightarrow$ A representative kitty eats too much.
- Output conjecture $\Longrightarrow$ She will get fatter.
- Secondary-stage output conjecture $\Longrightarrow$ She might get liver or heart disease.

Things can get more complicated quickly:

- Input statement $\Longrightarrow$ What if a representative kitty who eats too much is also more mobile and active?
- Output conjecture $\Longrightarrow$?

As economists, we want to know similar things about the economy and answer "what if" questions as we answer for the kitty.

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- If the model is getting larger and more complex, e.g., implicit models, then some of the tools may be hard to solve.
- We review and compare different approaches, e.g., exact hat algebra and other methods of counterfactual computation.
- We discuss an alternative computation method.
- We bridge exact hat algebra with other approach, then with the new method (when facing potential challenges).

What is CGE?

- It is a "Kitty-Cat" (counterfactual) model.
- Equilibrium with supply, demand and endogenous price mechanism (simultaneous equations).
- It can be comparative static or dynamic.
- We "know" the structural parameters.
- Mathematical modeling with computers.

Root of the CGE

1973 Press Photo: Wassily W. Leontief Nobel Prize Winner For Economics

Economic Models with No or Minimum Parameters

- Leontief went to Harvard and later worked at the BLS.
- He set up IO accounts for the US economy.
- He created a system that has all industries and private agents.
- After the project was commenced he started to realize that "**in addition to the table, he needed a model**" for the economy (Kohli, 2001; Mitra-Kahn, 2008).
- After WWII, BLS tasked to forecast steel demand.
- BLS took Leontief's IO estimates of the 1939 US economy and added demand and behavioral assumption.
- This model was one of the few models suggested that steel demand would not fall despite of the war.
- This prediction **turned out to be correct**.
- Leontief $\Longrightarrow$ Chenery $\Longrightarrow$ Johansen $\Longrightarrow$ Australia $\Longrightarrow$ Hertel.

Over the course of CGE history

Leontief's U.S. Economy $\Longrightarrow$ Johansen's Norway Economy

$\Longrightarrow$ Many CGE modelers' world economy

What **might** have been "changed":

- a bunch of theories $\Longrightarrow$ a bunch of computer codes
- richer input-output tables $\Longrightarrow$ more policy-oriented
- low computation capability $\Longrightarrow$ high computation capability
- "few" CGE models $\Longrightarrow$ "many" CGE models
- One school $\Longrightarrow$ "many" schools

Schools of Contemporary CGE

There are several classes or schools of CGE models:

- Australian School $\Longrightarrow$ GTAP School
- Colorado School
- Yale School (Cowles Foundation)

Today, everyone is doing similar counterfactual exercises, but they don't know each other **very well**.

- Australian School $\Longrightarrow$ Computation + Analytical GE
- Colorado School $\Longrightarrow$ Mixed Complementarity Problem
- Yale School $\Longrightarrow$ Exact Hat Algebra + Simultaneous Equations

Metaphysics or Science

- Economists study both things that are not *observed* variables in the universe, and utilize empirical observations and hard data.
- CGE modelers also study what is related to **change** across space, time, and different kinds of human beings.
- Meanwhile, data requirements are controlled through parsimonious assumptions (Balistreri and Hillberry, 2005)
- Some economists use simplified models to answer sophisticated questions, while some economists use sophisticated models to solve simple problems.
- Different schools of CGE modelers use different methods to deal with empirical challenges, and things we do not observe.
- A Penn State mathematician: why is $\sigma = 5$?
- Something I research at Yale: impacts of “rule-of-two” in the GTAP model $\Longrightarrow$ Are they shock-dependent?

DEK (Exact Hat Algebra)

Armington Model with Aggregate Trade

- We choose the CES utility function:

$$U_j = \left[\sum_i^n \beta_i^{\frac{1}{\sigma}} T_{ij}^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \tag{1}$$

- $\beta's$ are “shifters”, tastes, preferences.
- “Observed” in some counterfactual techniques, but “unobserved” in the real world.
- Trade share of i's goods in j is given by:

$$\pi_{ij} \equiv \frac{X_{ij}}{Y_j} = \beta_i \left(\frac{p_{ij}}{P_j} \right)^{1-\sigma} \tag{2}$$

- CES price index (unit expenditure function):

$$P_j = \left[\sum_i^n \beta_i p_{ij}^{1-\sigma} \right]^{\frac{1}{1-\sigma}} \tag{3}$$

Ask a "what-if" question

- We consider a counterfactual "what-if" scenario: policy instrument affecting trade costs: $\boldsymbol{\tau} \equiv [\tau_{ij}] \longrightarrow \boldsymbol{\tau'} \equiv [\tau'_{ij}]$
- For any generic variable V in the baseline equilibrium, V' is the value in the new equilibrium, and $\widehat{V} = V'/V$.
- The trade share is simplified as:

$$\pi_{ij} = \frac{\beta_i \tau_{ij}^{1-\sigma} FOB_i^{1-\sigma}}{\sum_k^n \beta_k \tau_{kj}^{1-\sigma} FOB_k^{1-\sigma}} \tag{4}$$

- Given the income definition:

$$Y_i = FOB_i e_i^0 \tag{5}$$

- We define $\delta_i \equiv \beta_i (e_i^0)^{\sigma-1}$, with e_i^0 which can be understood as i's endowment quantity.

Exact Hat Algebra

- The initial equilibrium is then given by:

$$\pi_{ij} = \frac{\delta_i(\tau_{ij} Y_i)^{1-\sigma}}{\sum_k^n \delta_k(\tau_{kj} Y_k)^{1-\sigma}} \tag{6}$$

- Note that Eq. (6) essentially gets rid of local price of endowment units which we do not have good price data for. δ_i is later also eliminated, so we do not need to estimate β_i.
- In counterfactual equilibrium:

$$\pi'_{ij} = \frac{\delta_i(\tau'_{ij} Y'_i)^{1-\sigma}}{\sum_k^n \delta_k(\tau_{kj} Y'_k)^{1-\sigma}} \tag{7}$$

- Eqs.(6) and (7) deliver

$$\hat{\pi}_{ij} = \frac{(\hat{\tau}_{ij} \hat{Y}_i)^{1-\sigma}}{\sum_k^n \pi_{kj}(\hat{\tau}_{kj} \hat{Y}_k)^{1-\sigma}} \tag{8}$$

Market-Clearing Condition

- The goods-market clearing conditions are

$$Y_i = \sum_{j}^{n} \pi_{ij} Y_j \tag{9}$$

$$Y_i' = \sum_{j}^{n} \pi_{ij}' Y_j' \tag{10}$$

- This gives:

$$\hat{Y}_i Y_i = \sum_{i}^{n} \hat{\pi}_{ij} \pi_{ij} \hat{Y}_j Y_j \tag{11}$$

Counterfactual Solution

- Substituting Eq.(8) in (11) gives us

$$\hat{Y}_i Y_i = \sum_{i}^{n} \frac{(\hat{\tau}_{ij}\hat{Y}_i)^{1-\sigma}}{\sum_k^n \pi_{kj}(\hat{\tau}_{kj}\hat{Y}_k)^{1-\sigma}} \pi_{ij}\hat{Y}_j Y_j \tag{12}$$

- Y_j and π_{ij} are data; $\hat{\tau}_{ij}$ are "what-if" known by us.
- Since there are N equations and N unknowns, we can solve the system above, finding $\hat{Y}_i$.
- Taking $\hat{Y}_i$ to Eq.(8), we can solve for $\hat{\pi}_{ij}$.
- Estimating σ without over-restricted normalization is difficult.
- From the econometric perspective, it is challenging to jointly identify U at initial equilibrium and σ.
- A common practice is to choose σ.

Closing GE Conditions

- Moving to solve for welfare change, we get rid of the local price term and use what we have from the data:

$$P_j^{1-\sigma} = \sum_i^n \beta_i p_{ij}^{1-\sigma} = \sum_i^n \delta_i (\tau_{ij} Y_i)^{1-\sigma} \tag{13}$$

- The welfare definition or income balance condition:

$$Y_j = U_j P_j \tag{14}$$

- We have country j's share of own consumption:

$$\pi_{jj} = \frac{\delta_j \tau_{jj}^{1-\sigma} Y_j^{1-\sigma}}{P_j^{1-\sigma}} = \delta_j \tau_{jj}^{1-\sigma} U_j^{1-\sigma} \tag{15}$$

ACR (2012) Result

- Therefore,

$$U_j = \left(\frac{\pi_{jj}}{\delta_j \tau_{jj}^{1-\sigma}} \right)^{\frac{1}{1-\sigma}}. \tag{16}$$

- δ_j is exogenous, $\tau_{jj} = 1$.
- Hence,

$$\hat{U}_j = (\hat{\pi}_{jj})^{\frac{1}{1-\sigma}}. \tag{17}$$

- The welfare prediction depends on **only two sufficient statistics:** (1) share of expenditure on domestic goods; (2) elasticity of substitution (and can be generalize to trade elasticity under other trade-cost specifications).
- This is, in fact, the **ACR result** (Arkolakis, Costinot and Rodríguez-Clare, 2012).

Mixed Complementarity Problems

Mixed Complementarity Problems

- The GE conditions can be formulated as mixed complementarity problems (MCP).
- The Mathiesen (1985) formulation of an Arrow-Debreu general equilibrium (Rutherford, 1995; Balistreri and Hillberry, 2005).
- Why MCP?
- Convenience of counterfactual calculations.
- Compact (Mathiesen, 1985) and transparent.
- The formulation allows corner solution (e.g., $q^* = 0$).

Mixed Complementarity Problems

- This approach requires us to first "know" β's (but not really).
- We also need to choose σ in order to estimate β's.
- The procedure identifies parameters at initial equilibrium by making the benchmark price normalization explicit:

$$FOB_i = 1 \quad \forall i \tag{18}$$

- This essentially chooses the units of local endowment e_i^0.
- From the exact hat algebra, we know that the price of endowment unit and quantity are irrelevant in determining counterfactual equilibrium.

Mixed Complementarity Problems

- This implies that the outcome of β's identified by any arbitrary choice of local price and endowment units does not affect counterfactual equilibrium (under the same GE specification).
- This can be verified computationally by shifting the FOB.
- Note that the numeraire price needs to be set consistently.
- Choosing the estimator, e.g., PPML, Least-Squares, subject to the same general equilibrium conditions.
- CES price index (unit expenditure function):

$$\left[\sum_{i}^{n} \beta_i (FOB_i \tau_{ij})^{1-\sigma}\right]^{\frac{1}{1-\sigma}} \geq P_j \quad \perp \quad U_j \geq 0$$

Mixed Complementarity Problems

- The explicit market-clearing condition can be solved by combining Eq. (2) and (9):

$$e_i^0 \geq \sum_j^n \left[\beta_i \frac{Y_j}{FOB_i} \left(\frac{FOB_i \tau_{ij}}{P_j} \right)^{1-\sigma} \right] \quad \perp \quad FOB_i \geq 0 \tag{19}$$

- The income balance condition:

$$U_j P_j \geq Y_j \quad \perp \quad P_j \geq 0 \tag{20}$$

- The income definition:

$$Y_i = FOB_i e_i^0 \tag{21}$$

Mixed Complementarity Problems

- The system formulated above can be solved in a non-linear program, such as NLP or mathematical programming with equilibrium constraints (MPEC).
- Fix the solved parameters, i.e., $\beta_i \longrightarrow$ data.
- Free income and price: data $\longrightarrow$ variables.
- This allows endogenous mechanism to determine counterfactual equilibrium.
- Implement policy instrument, i.e., $\tau_{ij} = 1 \longrightarrow \tau_{ij} = 1.25$.
- Identifying complementarity variables.
- Solving for counterfactual equilibrium.

EMCP versus Exact Hat

- We may denote the full computation procedure as an EMCP approach (Estimation and MCP).
- Using Anderson and van Wincoop (2003) data as an exercise.
- It can be verified that the results solved using the EMCP are equivalent to the ACR and exact hat results.
- Both methods "ignore" β's, benchmark prices and endowment quantities in some way.
- Exact hat elegantly eliminates β's and prices directly, but one has to derive the counterfactual formulae.
- This can be quite tedious, and sometimes challenging if choosing more flexible models.
- EMCP approach does "estimate" β's, but does so by choosing the units and prices.
- The choice of prices, e.g., setting to unity, does impact the outcome of $\beta's$, but not the counterfactual equilibrium.

“Estibration”

- Coined by Balistreri and Hillberry (2005): estib.zip.
- This approach has some sense of “estibration”, while not fully estibrate, but does “calibrate” model parameters to one benchmark point, by estimating them using benchmark data and the choice of other **exogenous** information, that is not readily accessible, such that the general equilibrium system is fully operationalized.
- Because the estimation does all the hard work, we do not have to algebraically invert the model to calibrate the function coefficients as some work done in the CGE literature, e.g., “calibrated share form”, Rutherford (1995).
- Both EMCP has some challenges, when choosing more flexible models. Implicit models are some good examples.

Similarities and Issues

- DEK and CGE are both counterfactual methods.
- Again, in an Armington context, DEK eliminates price data and β shifters explicitly, while CGE eliminates price data and β shifters implicitly.
- The two methods yield the same ACR welfare result, but requires one to fix σ. The size of welfare thus depends on σ:

$$\hat{U}_j = (\hat{\pi}_{jj})^{\frac{1}{1-\sigma}}.$$

- In general, both literatures "suggest" that shifters are not particularly interesting, but without claiming that the σ would also **empirically** depend on the shifters when fitting/calibrating the model.
- In some sense, both Kortumese and CGE models are "discrete-choice" models, but trade economists have never estimated σ as done in IO literature.
- Likewise, jumping out of the box, Berry (1994) and EK (2002) delineate shifters in *similarly different* ways.

On-going Empirical Work

- We can't estimate σ in a constrained optimization without normalizing utility.
- Number of structural parameters > Degree of freedom (data)
- In constrained optimization, there are not enough normalizing constants under the CES assumption (to remove the DF par).
- In a more general model where I estimate σ_i, the objective essentially takes "shares" information, and the procedure estimates utility and other model parameters simultaneously.
- This is conceptually similar to Berry (1994).
- Therefore, we derived an empirical equation following Berry (1994) with IV approach.
- We find that the selected **instrument** that we use in econometrics is conceptually related to the "closure" condition in some of the CGE models.

An Alternative Method

Implicit and Indirect Demand

- Derivation of implicit models using exact hat can be difficult:

$$G = \sum_i \beta_i U_j^{e_i(1-\sigma_i)} (\frac{L_j FOB_i}{Y_j})^{1-\sigma_i} \equiv 1. \tag{22}$$

- In this case, Hanoch (1975), utility cannot be isolated and there is larger parameter space.
- Furthermore, it is difficult to identify complementarity variables because, for example, one single derived functional form represents both income balance condition and unit expenditure function:

$$P_j = \frac{\left[\sum_i \beta_i U_j^{e_i - e_i\sigma_i - 1} (1-\sigma_i) FOB_i^{1-\sigma_i} (Y_j/L_j)^{\sigma_i - 1} e_i \right]}{\left[\sum_k \beta_k U_j^{e_k(1-\sigma_k)} (1-\sigma_k) FOB_k^{1-\sigma_k} (Y_j/L_j)^{\sigma_k - 2} \right]} \tag{23}$$

Ambiguity in Inequality Constraints

- If we parameterize the system to a CES function, then Eq. (23) will collapse to the income balance condition, but not the unit expenditure function.
- This also makes it puzzling to solve using the conventional MCP approach.
- Complementarity variable is ambiguous.
- Alternatively, I use an estimation approach (e.g., based on the MPEC algorithm), and computes for both model parameters and counterfactuals as in the EMCP.

An Alternative Method

- The MPEC is efficient; see Su and Judd (2012); and Pakes, Ostrovsky, and Berry (2007).
- Methodologically, standard NLP will also work (at interior).
- It takes two solves, one to “estibrate” model parameters that are identified by the price normalization scheme and the data, to the benchmark equilibrium, while other parameters identified invariant to normalizing constants.
- The second solve takes care of the counterfactual calculation.

An Alternative Method

- The MPEC embeds an MCP, so it is suitable to estimate the general equilibrium relationship.
- It solves the problem by treating a parametric derived from the NLP problem fixed, while setting initial benchmark by constraining the likelihood of the objective function, such that $\overline{\mathscr{J}}(\overline{\mathscr{F}}, \overline{\boldsymbol{\alpha}}, \overline{\boldsymbol{e}}) \equiv \mathscr{J}^0(\overline{\mathscr{F}}, \overline{\boldsymbol{\alpha}}, \overline{\boldsymbol{e}})$ with the choice of numeraire.
- $\overline{\mathscr{J}}$ is the estimated value of the objective function.
- While the complementarity theory is a discipline of mathematical optimization, the MCP does not carry a visible objective function.
- Nonetheless, shifting independent variables before and after the counterfactual equilibrium in fitting to the same data on dependent variable must have different likelihood values.

An Alternative Method

- The procedure essentially calibrates to the benchmark model by calibrating to the likelihood at initial equilibrium.
- In the second solve, release its restriction to the likelihood, while freeing model variables, compute the counterfactual equilibrium directly after the exogenous shocks.
- $\mathscr{J}^0(\overline{\mathscr{F}}, \overline{\boldsymbol{\alpha}}, \overline{\boldsymbol{e}}) \longrightarrow \mathscr{J}^1(\mathscr{F}, \overline{\boldsymbol{\alpha}}, \overline{\boldsymbol{e}})$
- data $\longrightarrow$ variables
- $\boldsymbol{\tau} \equiv [\tau_{ij}] \longrightarrow \boldsymbol{\tau'} \equiv [\tau'_{ij}]$
- It can be shown that results are equivalent to MCP and DEK.

Concluding Remarks

- Why do we want to do this?
- Economic models are becoming more and more complex.
- Computational expense, however, is becoming lower.
- Not every model is (easily) solvable by hand.
- Not every model has a clear intuition on complementarities.
- Not everyone needs to "fit the data to (large) models".
- It should really be the other way around.
- In this approach, one only needs to know the primal functional form of GE conditions.
- Other approaches rely largely on computations anyway.
- If we can show that the results with relatively less efforts can produce the same results, and are consistent with the theory, we should consider using it.